Chapter 13. Gravitational Mirages

- *How can Newton's mechanics predict the deflection of light by the Sun?*
- *Does Einstein predict a different value of light deflection than Newton? If so, which prediction do we observe?*
- *Does the amount of deflection depend on the energy/wavelength of the light?*
- *Can a center of gravitational attraction act like a lens? Can it create a mirage?*
- *Can a gravitational lens magnify distant objects?*
- *Can a planet around a distant star act as a mini-gravitational lens?*
- *How can we use light deflection to detect and measure mass that we cannot see?*

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CHAPTER 13

Gravitational Mirages

Edmund Bertschinger & Edwin F. Taylor *

Einstein was discussing some problems with me in his study when he suddenly interrupted his explanation and handed me a cable from the windowsill with the words, "This may interest you." It was the news from Eddington [actually from Lorentz] *confirming the deviation of light rays near the sun that had been observed during the eclipse. I exclaimed enthusiastically, "How wonderful, this is almost what you calculated." He was quite unperturbed. "I knew that the theory was correct. Did you doubt it?" When I said, "Of course not, but what would you have said if there had not been such a confirmation?" he retorted, "Then I would have to be sorry for dear God. The theory is correct." ["Da könnt' mir halt der liebe Gott leid tun. Die Theorie stimmt doch."]*

—Ilse Rosenthal-Schneider

13.1 ■ GRAVITY TURNS STARS AND GALAXIES INTO LENSES

Euclid overthrown

1919: Einstein's prediction of light deflection verified.

Arthur Eddington's verification of Albert Einstein's predicted deflection of starlight by the Sun during the solar eclipse of 1919 made Einstein an instant celebrity, because this apparently straightforward observation replaced Newton's two-centuries-old mechanics and Euclid's twenty-two-centuries-old geometry with Einstein's revolutionary new general relativity theory.

Long history of deflection predictions

Einstein's predicted deflection of light by a star has a long history, traced out in the following timeline. His prediction also implied that a gravitating structure can act as a lens, bending rays around its edge to concentrate the light and even to form one or more distorted images. We call the result **gravitational lensing**. Gravitational lensing has grown to become a major tool of modern astronomy.

*Draft of Second Edition of *Exploring Black Holes: Introduction to General Relativity*

Timeline: Deflection of Starlight

Timeline of starlight deflection

History of predicting, discovering, and employing the deflection of starlight for cosmological research:

1801 Johann George von Soldner makes a Newtonian calculation of the deflection of starlight by the Sun (Section 13.2). He predicts a deflection half as great as Einstein later derives and observation verifies. Soldner's predicted result, 0.84 arcsecond, was not followed up by astronomers. (One arcsecond is 1/3600th of one degree.)

1911 Einstein recalculates Soldner's result (obtaining 0.83 arcsecond) without knowing about Soldner's earlier work.

1914 Einstein moves to Berlin and asks astronomer Erwin Freundlich if the tiny predicted result can be measured. At dinner, Einstein covers Mrs. Freundlich's prize table cloth with equations. She later laments, "Had I kept it unwashed as my husband told me, it would be worth a fortune." Freundlich points out that the measurement Einstein seeks can be made during a total solar eclipse predicted for the Crimea in Russia on August 21, 1914. Freundlich organizes an expedition to that location. World War I breaks out between Germany and Russia on August 1; Freundlich and his team are arrested as spies, so cannot make the observation. (They are quickly exchanged for Russian prisoners; the sky is cloudy anyway.)

1915 In November Einstein applies the space curvature required by his theory (Section 13.5) to recalculate light deflection by the Sun, finds 1.7 arcsecond, double the previous value. A week later Einstein completes the logical structure of his theory of general relativity. (In January 1916 astronomer Karl Schwarzschild, serving as a German artillery officer on the Russian front, finds a solution to Einstein's equations—the Schwarzschild metric—for a spherically symmetric center of attraction.)

1919 Arthur Eddington leads a post-war group to the island of Principe, off the coast of West Africa, to measure starlight deflection during a total eclipse. He reports a result that verifies Einstein's prediction. "Lights All Askew in the Heavens" headlines the *New York Times* (Figure 1). Later observations, using both light and radio waves, validate Einstein's prediction to high accuracy.

1936 R. W. Mandl, a German engineer and amateur astronomer, asks Einstein if the chance alignment of two stars could produce a ring of light from gravitational deflection. Einstein writes it up for the journal *Science* and remarks condescendingly to the editor, "Thank you for your cooperation with the little publication which Herr Mandl squeezed out of me. It is of little value, but it makes the poor guy happy." In the paper Einstein says, "Of course, there is no hope of observing this phenomenon directly."

1937 Fritz Zwicky, controversial Swiss-American astronomer, says that Einstein is wrong, because galaxies can produce observable deflection of light. He also anticipates the use of gravitational lenses to measure the mass of galaxies.

1979 Forty-two years after Zwicky's insight, the first gravitational lens is discovered and analyzed (Figure 14).

NOW: Gravitational lensing becomes a major tool of astronomers and astrophysicists.

Gravitational lens

Gravitational mirage

In the present chapter we re-derive Soldner's expression for deflection of starlight by the Sun, re-derive Einstein's general-relativistic prediction, then apply results to any astronomical object, such as a galaxy, that acts as a **gravitational lens**. This lens deflects rays from a distant source to form a distorted image for an observer on the opposite side of, and distant from, the lensing object. A gravitational lens can yield multiple images, arcs, or rings. It can also magnify distant objects. We adopt the descriptive French term for such a distorted image: **gravitational mirage**. Astronomers use gravitational mirages to study fundamental components of the Universe, such as the presence and abundance of dark matter, and to detect planets orbiting around distant stars.

Several features of applied gravitational lensing simplify our study of gravitational mirages:

Simplifying conditions

1. The source of light is a long way from the deflecting structure.
2. The observer is a long way from (and on the opposite side of) the deflecting structure.
3. Light ray deflection by ordinary stars and galaxies is *very* small. The Sun's maximum deflection of starlight is 1.75 arcsecond = 1.75/3600 degree = 0.000486 degree. Multiple rays may connect emitter and observer, but we are safe in treating every deflection as very small.

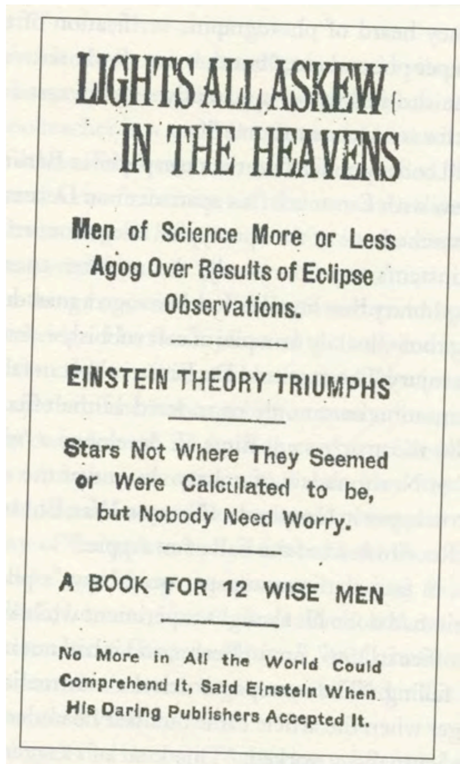
13.2 ■ NEWTONIAN STARLIGHT DEFLECTION (SOLDNER)

Pretend that a stone moves with the speed of light.

Newtonian analysis of light deflection

In 1801 Johann von Soldner extended Newton's particle mechanics to a "particle of light" in order to predict its deflection by the Sun. His basic idea was to treat light as a very fast Newtonian particle. Soldner's analysis yields an incorrect prediction. Why do we repeat an out-of-date Newtonian analysis of light deflection? Because the result highlights the radical revolution that Einstein's theory brought to spacetime (Section 13.5). However, we do not follow Soldner's original analysis, but adapt one by Joshua Winn.

Suppose the particle of light moves in the x direction tangentially past the attracting object (Figure 2). We want to know the y-component of velocity that this "fast Newtonian particle" picks up during its passage. Integrate Newton's second law to determine the change in y-momentum. We use



LIGHTS ALL ASKEW IN THE HEAVENS

Men of Science More or Less Agog Over Results of Eclipse Observations.

EINSTEIN THEORY TRIUMPHS

Stars Not Where They Seemed or Were Calculated to be, but Nobody Need Worry.

A BOOK FOR 12 WISE MEN

No More in All the World Could Comprehend It, Said Einstein When His Daring Publishers Accepted It.

FIGURE 1 Headline in the *New York Times* November 10, 1919. The phrase "12 wise men" refers to the total number of people reported to understand general relativity at that time.

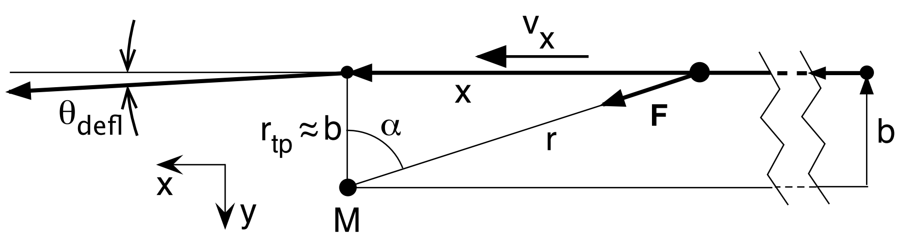


FIGURE 2 Symbols for the Newtonian calculation of the deflection of light that treats a photon as a very fast particle. This approximation assumes that the deflection is very small and occurs suddenly at the turning point r_{tp}. Not to scale.

conventional units in order to include the speed of light c explicitly in the analysis.

$$\int_{-\infty}^{+\infty} F_{\text{y}} dt = \Delta p_{\text{y}} = m_{\text{kg}} \Delta v_{\text{y}} \qquad \text{(Newton)} \qquad (1)$$

The y-component of the gravitational force on the particle is:

$$F_{\rm y} = F\cos\alpha = \frac{GM_{\rm kg}m_{\rm kg}}{r^2}\cos\alpha \qquad \text{(Newton, conventional units)} \qquad (2)$$

Assume that the speed of the "particle" is the speed of light: $v = (v_{\rm x}^2 + v_{\rm y}^2)^{1/2} = c$. We expect the deflection to be extremely small, $v_{\rm y} \ll c$, so take $v_{\rm x} \approx c$ to be constant during the deflection. From Figure 2:

$$\frac{b}{r} = \cos\alpha \quad \text{so that} \quad \frac{1}{r^2} = \frac{\cos^2\alpha}{b^2} \quad \text{and} \qquad \text{(Newton)} \qquad (3)$$

$$dt = \frac{dx}{v_{\rm x}} \approx \frac{dx}{c} \quad \text{and} \quad x = b\tan\alpha \quad \text{so} \quad dx = \frac{b\,d\alpha}{\cos^2\alpha} \qquad \text{(Newton)} \qquad (4)$$

As the particle flies past the center of attraction, the angle α swings from $-\pi/2$ to (slightly more than) $+\pi/2$. Substitute from equations (2) through (4) into (1), cancel the "Newtonian photon mass $m_{\rm kg}$" from both sides of the resulting equation, and integrate the result:

$$\frac{GM_{\rm kg}}{bc}\int_{-\pi/2}^{+\pi/2}\cos\alpha\,d\alpha = \frac{2GM_{\rm kg}}{bc} = \Delta v_{\rm y} \quad \text{(Newton, conventional units)} \quad (5)$$

The integral in (5) has the value 2. Because the deflection angle $\theta_{\rm defl}$ is very small, we write:

$$\theta_{\rm defl} \approx \frac{\Delta v_{\rm y}}{c} = \frac{2}{b}\left(\frac{GM_{\rm kg}}{c^2}\right) \rightarrow \frac{2M}{b} \approx \frac{2M}{r_{\rm tp}} \qquad \text{(Newton)} \qquad (6)$$

The next-to-last step in (6) reintroduces mass in units of meters from equation (10) in Section 3.2. The last step, which equates the turning point $r_{\rm tp}$ with impact parameter b, follows from the tiny value of the deflection, as spelled out in equation (12).

Apply (6) to deflection by the Sun. The smallest possible turning point $r_{\rm tp}$ is the Sun's radius, leading to a maximum deflection:

$$\theta_{\rm defl,max} \approx \frac{2M}{r_{\rm Sun}} = 4.25\times10^{-6}\,\text{radian} = 2.44\times10^{-4}\,\text{degree} \quad \text{(Newton)} \ (7)$$

Soldner's predicted deflection

Multiply this result by 3600 arcseconds/degree to find the maximum deflection $\theta_{\rm defl,max} \approx 0.877$ arcsecond. (Soldner predicted 0.84 arcsecond.) Section 13.4 shows that the correct prediction—verified by observation—is twice as large: $\theta_{\rm defl,max} \approx 4M/r_{\rm Sun} = 1.75$ arcseconds. (Einstein predicted 1.7 arcseconds.) All of these predicted and observed deflection angles are much, much smaller than even the tiny angle $\theta_{\rm defl}$ shown in Figure 2.

Both Soldner's result (6) and Einstein's—equation (18) in Box 1—tell us that the deflection angle is inversely proportional to the turning point $r_{\rm tp}$. That is, maximum bending occurs for a ray that passes closest to the deflecting object (right panel, Figure 3). Contrast this with the conventional glass *optical*

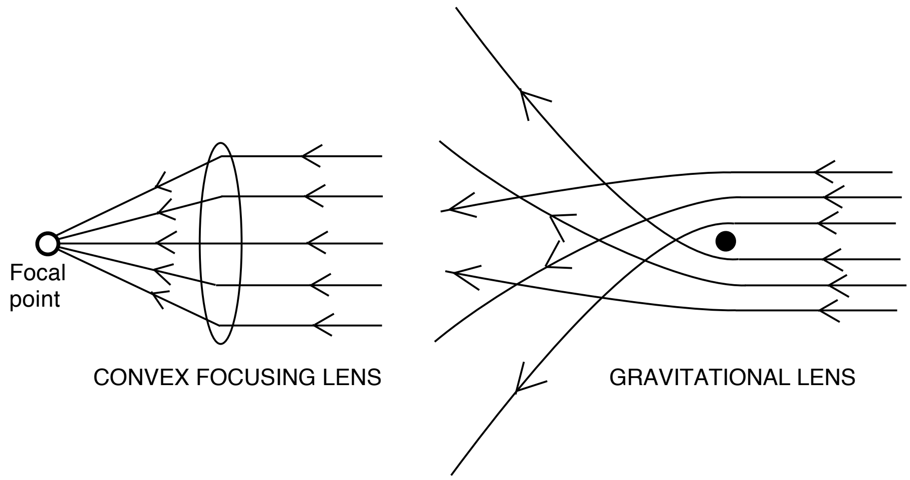


FIGURE 3 Schematic comparison of deflection of light by glass lens and gravitational lens. **Left panel:** In a conventional convex glass focusing lens, deflection increases farther from the center in such as way that incoming parallel rays converge to a *focal point*. **Right panel:** Deflection by a gravitational lens is greatest for rays that pass closest to the center—equation (6). *Result:* no focal point, which guarantees image distortion by a gravitational lens.

A gravitational lens *must* distort the image.

focusing lens, such as the lens in a magnifying glass, whose edge bends light more than its center (left panel, Figure 3). Parallel rays incident on a good optical lens converge to a single point, called the *focal point*. The existence of a focal point leads to an undistorted image. A gravitational lens—with maximum deflection for the closest ray—does not have a focal point, which results in image distortion. The base of a stem wineglass acts similarly to a gravitational lens, and shows some of the same distortions (Figure 4).

QUERY 1. Quick Newtonian analysis

Show that you obtain the same Newtonian result (6) if you assume (Figure 5) that transverse acceleration takes place only across a portion of the trajectory equal to twice the turning point and that this transverse acceleration is uniform downward and equal to the acceleration at the turning point.

13.3 ■ LIGHT DEFLECTION ACCORDING TO EINSTEIN AFTER 1915

In from infinity, out to infinity

Analysis of large deflection

Now we analyze the deflection of starlight by a center of attraction predicted by general relativity. Any incoming light with impact parameter $|b| > b_{\text{critical}}$ does not cross the event horizon, but rather escapes to infinity (Figure 3 in Section 12.3). Figure 6 shows resulting rays from the star lying at map angle $\phi_\infty = 0$ with positive impact parameter b. This light approaches the black hole

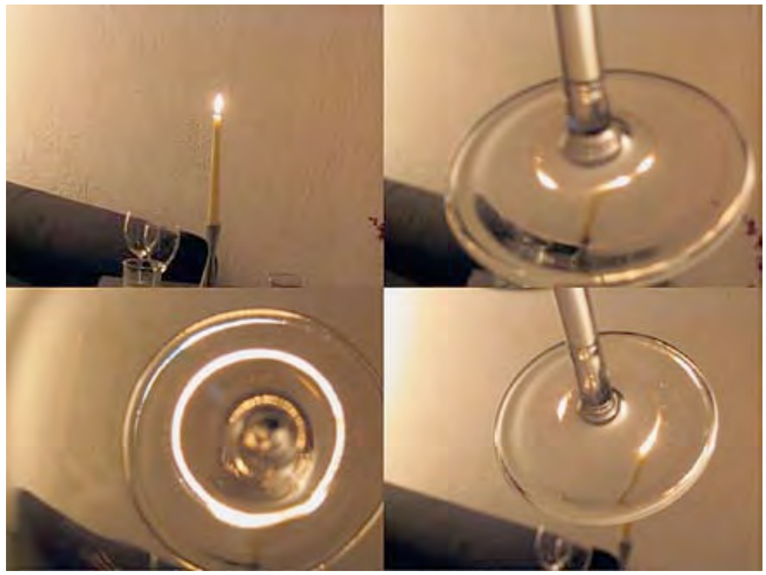


FIGURE 4 The base of a stem wine glass has lensing properties similar to that of a gravitational lens. The source of light, at the top left, is a candle. Tilting the wine glass base at different angles with respect to the source produces multiple images similar to those seen in gravitational mirages. The bottom right panel shows a double image and the top right four images. The bottom left—looking down the optical axis of the wine glass—shows an analogy to the full *Einstein ring* (Figure 12). Images courtesy of Phil Marshall.

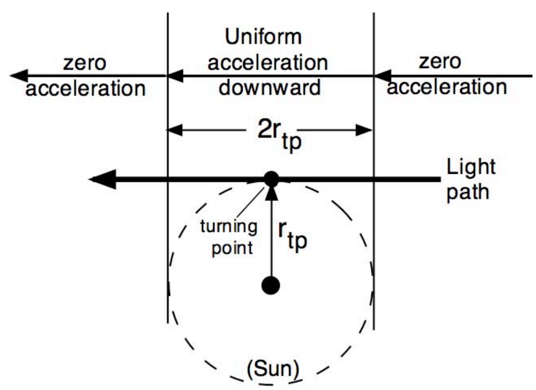


FIGURE 5 Figure for Query 1: Alternative derivation of Newtonian light deflection. Dashed circle: outline of our Sun, with light ray skimming past its edge, so that its turning point $r_{\rm tp}$ equals the Sun's radius $r_{\rm Sun}$.

from a distant source, deflects near the black hole, then recedes from the black hole to be seen by a distant observer. The ray is symmetric on the two sides of the turning point, so the total change in direction along the ray is twice the

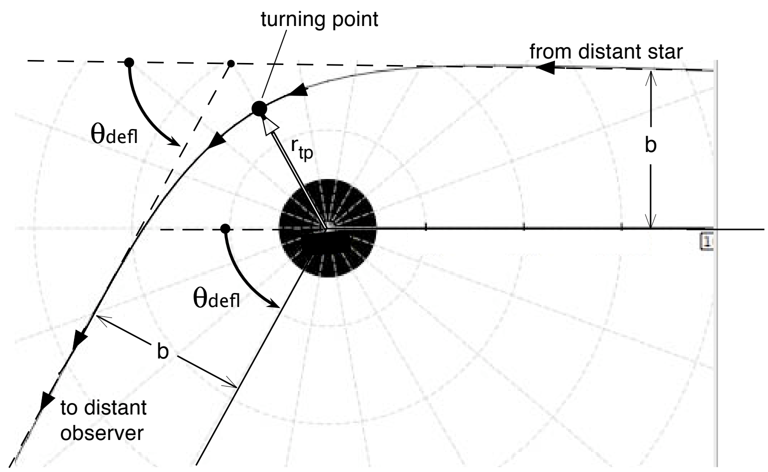


FIGURE 6 Total deflection $\theta_{\rm defl}$ of a ray with impact parameter b that originates at a distant star. This ray deflects near the center of attraction, then runs outward to a distant observer. Deflection $\theta_{\rm defl}$, is the *change* in direction of motion of the flash. As usual, the positive direction of rotation is counterclockwise.

change in direction between the turning point and either the distant source or the distant observer.

Reach back into Chapter 11 for the relevant equations. Start with equation (40) of Section 11.6, which applies to an observer *after* the turning point. In the present situation, the map angle of the distant star is $\phi_\infty = 0$, the observer is far from the center of attraction, so $r_{\rm obs} \rightarrow \infty$, and from the definition of the deflection angle, $\phi_{\rm obs} = \pi + \theta_{\rm defl}$. Substitute these into equation (40) of Section 11.6 to obtain:

$$\theta_{\rm defl} + \pi = 2\int_{r_{\rm tp}}^{\infty} \frac{b}{r^2}\left[1 - \frac{b^2}{r^2}\left(1 - \frac{2M}{r}\right)\right]^{-1/2} dr \qquad (8)$$

The integrand in (8) is a function of b, and the lower integration limit is $r_{\rm tp}$. To remove these complications, make the substitution

$$u \equiv \frac{r_{\rm tp}}{r} \quad \text{so that} \quad dr = -\frac{r_{\rm tp}}{u^2}du \qquad (9)$$

where $r_{\rm tp}$ and b are constants. The variable u has the value $u = 0$ at the distant star and the value $u = 1$ at the turning point. With substitutions (9), equation (8) becomes:

$$\theta_{\rm defl} + \pi = \frac{2b}{r_{\rm tp}}\int_0^1 \left[1 - \left(\frac{b}{r_{\rm tp}}\right)^2 u^2\left(1 - \frac{2Mu}{r_{\rm tp}}\right)\right]^{-1/2} du \qquad (10)$$

Both b and $r_{\rm tp}$ are parameters (constants) in the integrand of (10). Use equation (27) in Section 11.4 to convert $r_{\rm tp}$ to b. Then b is the only parameter

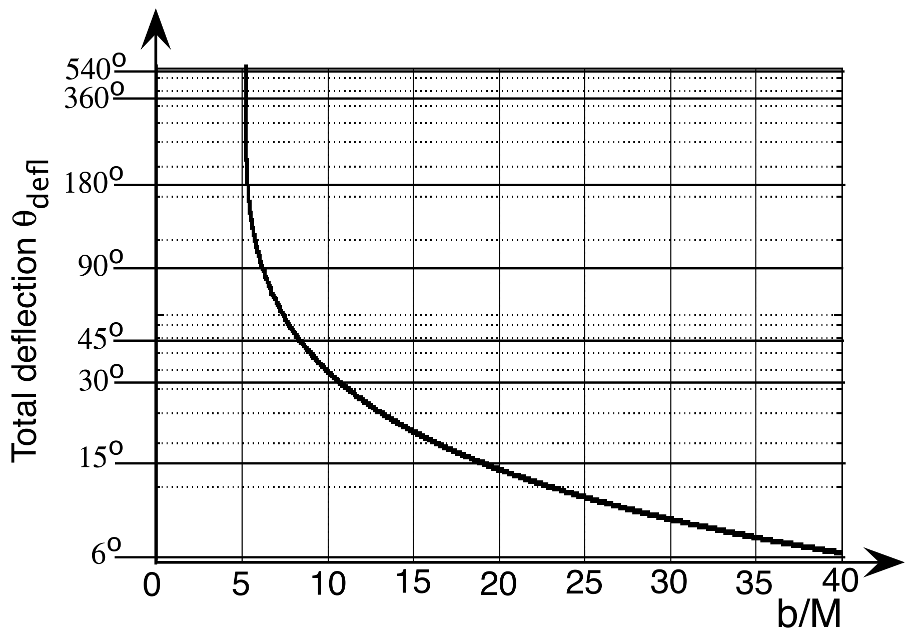


FIGURE 7 Map deflection θ_{defl} as a function of positive values of b from the numerical integration of (10). The vertical scale is logarithmic, which allows display of both small and large values of deflection.

in (10). Figure 7 plots results of a numerical integration for positive values of b. The magnitude of θ_{defl} covers a wide range; the semi-log plot makes it easier to read both small and large values of this angle.

Comment 1. Infinite deflection?
Why does the total deflection in Figure 7 appear to increase without limit as the impact parameter b drops to a value close to five? In answer, look at Figure 1 in Section 11.2. When the impact parameter of an approaching ray takes on the value $b_{\text{critical}} = 3(3)^{1/2} = 5.196$, then the ray goes into a knife-edge orbit at $r = 3M$. In effect the deflection angle becomes infinite, which is consistent with the plot in Figure 7.

13.4 ■ LIGHT DEFLECTION THROUGH SMALL ANGLES

Einstein's prediction

Simplification for small deflection

Equation (10) becomes much simpler when the deflection is very small, that is when r_{tp} of the turning point is much larger than $2M$. When a ray passes our Sun, for example, the r-value of its turning point must be greater than or equal to the Sun's radius r_{Sun} if the ray is to make it past the Sun at all. Box 1 approximates the deflection of a light ray with turning point $r_{\text{tp}} \gg 2M$.

Box 1. Starlight Deflection: Small-Angle Approximation

We seek an approximate expression for deflection $\theta_{\rm defl}$ in equation (10) when a ray passes sufficiently far from the center of attraction to satisfy the condition:

$$\frac{2M}{r_{\rm tp}} \equiv \epsilon \ll 1 \qquad (11)$$

Equation (7) reminds us that the maximum deflection by our Sun is very small: $\epsilon_{\rm Sun} = 4.253 \times 10^{-6}$ radian. In the following we make repeated use of the first order approximation inside the front cover. From (11) plus equation (27) in Section 11.4 for the turning point,

$$\frac{b}{r_{\rm tp}} = \left(1 - \frac{2M}{r_{\rm tp}}\right)^{-1/2} = (1-\epsilon)^{-1/2} \approx 1 + \frac{\epsilon}{2} \qquad (12)$$

Then approximate one expression in the integrand of (10) as:

$$\begin{aligned} &\left(\frac{b}{r_{\rm tp}}\right)^2 u^2 \left(1 - \frac{2Mu}{r_{\rm tp}}\right) \qquad (13)\\ &\approx \left(1+\frac{\epsilon}{2}\right)^2 u^2\,(1-u\epsilon)\\ &\approx (1+\epsilon)u^2(1-u\epsilon)\\ &\approx u^2 + (1-u)u^2\epsilon \end{aligned}$$

At each step we neglect terms in ϵ^2. With this substitution, the square bracket expression in (10) becomes

$$\begin{aligned} &\left[1-u^2-(1-u)u^2\epsilon\right]^{-1/2} \qquad (14)\\ &= (1-u^2)^{-1/2}\left[1 - \frac{(1-u)}{1-u^2}u^2\epsilon\right]^{-1/2}\\ &= (1-u^2)^{-1/2}\left[1 - \frac{u^2\epsilon}{1+u}\right]^{-1/2}\\ &\approx \frac{1}{(1-u^2)^{1/2}}\left[1 + \frac{u^2\epsilon}{2(1+u)}\right] \end{aligned}$$

The coefficient of the integral in (10) is $2b/r_{\rm tp} = 2 + \epsilon$ from (12). Combine this with the last line of (14) to find the expressions that we want to integrate, again to first order in ϵ.

$$\frac{2+\epsilon}{(1-u^2)^{1/2}} + \frac{u^2\epsilon}{(1+u)(1-u^2)^{1/2}} \qquad (15)$$

From a table of integrals:

$$\int_0^1 \frac{du}{(1-u^2)^{1/2}} = \frac{\pi}{2} \qquad (16)$$

This occurs twice in (15), one integral multiplied by 2, the other by ϵ. The integral of the second term in (15) becomes:

$$\int_0^1 \frac{u^2 du}{(1+u)(1-u^2)^{1/2}} = 2 - \frac{\pi}{2} \qquad (17)$$

In (15) this is multiplied by ϵ. Combine the results of (15) through (17) to write down the expression for $\theta_{\rm defl}$ in (10), not forgetting the term π on the left side, which cancels an equal term on the right side:

$$\theta_{\rm defl} \approx 2\epsilon \approx \frac{4M}{r_{\rm tp}} \ll 1 \qquad (18)$$

where $r_{\rm tp}$ is the turning point, the r-value of closest approach. Equation (18) is Einstein's general relativistic prediction for deflection when (11) is satisfied. Section 13.2 showed that Soldner's Newtonian analysis predicts a value of the light deflection half as great as (18).

Important note: Our derivation of (18) assumes that the deflecting structure is either (a) a point—or a spherically symmetric object—with $r_{\rm tp} \geq$ the r-value of this structure, or (b) a black hole approached by light with impact parameter $b/M \gg 1$. It is not valid when the the light passes close to a non-spherical body, like a galaxy.

Deflection when $R \gg M$

From the result of Box 1 we predict that the largest deflection of starlight by the Sun occurs when the light ray skims past the edge of the Sun. From (18), this maximum deflection is:

$$\begin{aligned} \theta_{\rm defl,max} = \frac{4M}{r_{\rm Sun}} &= 8.49 \times 10^{-6} \text{ radian} \qquad (19)\\ &= 4.87 \times 10^{-4} \text{ degree}\\ &= 1.75 \text{ arcsecond} \end{aligned}$$

QUERY 2. Value of $r_{\rm tp}/M$ for various deflections

Compute eight values of the turning point $r_{\rm tp}$, namely the four deflection angles of Items A through D below for each of two cases. *Case I:* The mass of a star like our Sun. *Case II:* The total mass of the visible stars in a galaxy, approximately 10^{11} Sun masses.

A. $\theta_{\rm defl} \approx$ one degree
B. $\theta_{\rm defl} \approx$ one arcsecond
C. $\theta_{\rm defl} \approx 10^{-3}$ arcsecond
D. $\theta_{\rm defl} \approx 10^{-6}$ arcsecond
E. For Case I, compare the values of $r_{\rm tp}$ with the r-value of the Sun.
F. For Case II, compare the values of $r_{\rm tp}$ with the r-value of a typical galaxy, 30 000 light years.
G. Which cases can occur that lead to a deflection of one arcsecond? 10 arcseconds? Are these turning points inside the radius of a typical galaxy? If so, we cannot correctly use deflection equation (18), which was derived for either a point lens or for a turning point $r_{\rm tp}$ outside a spherically symmetric lens.

Starting with Section 13.6, the remainder of this chapter describes multiple uses of the single deflection equation (18) in astronomy, astrophysics, and cosmology. But first we take a detour to outline the profound revolution that Einstein's prediction of the Sun's deflection of starlight made in our understanding of spacetime geometry.

13.5 ■ DETOUR: EINSTEIN DISCOVERS SPACE CURVATURE

Einstein discovers a factor of two and topples Euclid.

Einstein's initial error in light deflection

Here we take a detour to show how Einstein, in effect, used an incomplete version of the Schwarzschild metric to make an incorrect prediction of the Sun's gravitational deflection of starlight, a prediction that he himself corrected before observation could prove him wrong. His original misconception was to pay attention to spacetime curvature embodied in the t-coordinate, but not to realize at first that the r-coordinate is also affected by spacetime curvature.

Global gravitational potential

In 1911, as he developed general relativity, Einstein predicted the deflection of starlight that reaches us by passing close to the Sun. Einstein recognized that in Newtonian mechanics the expression $-M/r$ is potential energy per unit mass, called the **gravitational potential**, symbolized by Φ. Einstein's initial metric generalized the gravitational potential of Newton around a point mass M:

$$\Phi = -\frac{M}{r} \qquad \text{(Newton)} \tag{20}$$

Einstein initially warped only t-coordinate.

Einstein's 1911 analysis was equivalent to adopting a global metric that we now recognize as incomplete:

$$d\tau^2 = (1+2\Phi)\,dt^2 - dr^2 - r^2 d\phi^2 = \left(1 - \frac{2M}{r}\right) dt^2 - ds^2 \qquad (21)$$

("half-way to Schwarzschild")

"Half way to Schwarzschild"

The r, ϕ part of metric (21) is flat, as witnessed by the Euclidean expression $ds^2 = dr^2 + r^2 d\phi^2$. In contrast, the r-dependent coefficient of dt^2 shows that the t-coordinate has a position-dependent warpage. Thus metric (21) is "half way to Schwarzschild," even though in 1911 Einstein did not yet appreciate the centrality of the metric, and the derivation of the Schwarzschild metric was almost five years in the future.

For light, set $d\tau = 0$ in (21), which then predicts that the map speed of light decreases as it approaches the Sun:

$$\left|\frac{ds}{dt}\right|_{\text{light}} = \left(1 - \frac{2M}{r}\right)^{1/2} \approx 1 - \frac{M}{r} \quad (M/r \ll 1) \quad \text{(Einstein 1911)} \qquad (22)$$

Equation (22) reduces the problem of light deflection to the following exercise in geometric optics: "Light passes through a medium in which its speed varies with position according to equation (22). Use Fermat's Principle to find a light ray that grazes the edge of the Sun as it travels between a distant source and a distant observer on opposite sides of the Sun."

Fermat's principle

Fermat's Principle, derived from standard classical electromagnetic theory of light, says that light moves along a trajectory that minimizes the total time of transit from source to observer. (This is classical physics, so space is flat, as (21) assumes.) Einstein used Fermat's Principle—geometric optics plus equation (22)—to calculate a deflection, $\theta_{\text{defl}} = 2M/r_{\text{tp}}$, equal to half of the observed value.

Gravity also warps r-coordinate.

In his initial prediction, however, Einstein failed to understand that gravity also curves the r, ϕ part of spacetime near a center of attraction. We can now see this initial error and correct it ourselves. The Schwarzschild metric (3.5) shows that the r, ϕ portion of the metric is not flat; the term dr^2 in (21) should be $dr^2/(1 - 2M/r)$. The difference arises from the contribution of the $dr, d\phi$ components to curvature.

We can derive the radial component of map light velocity from the correct Schwarzschild metric:

$$\left|\frac{dr}{dt}\right|_{\text{light}} = 1 - \frac{2M}{r} \qquad \text{(radial motion, Schwarzschild)} \qquad (23)$$

Radial motion of light further slowed by space curvature.

Compare the results of equations (22) and (23). Light that grazes the surface of the Sun in its trajectory between a distant star and our eye travels *almost* radially during its approach to and recession from the Sun. Fermat's Principle still applies, but the angle of deflection predicted from the change in coordinate speed of light in (23) is twice that of the preliminary prediction derived from (22).

Profound insight: Space also curved

Einstein's realization that the r, ϕ part of global coordinates must be curved, along with the t part, was a profound shift in understanding, from which his field equations emerged. Einstein's doubled prediction of light deflection was tested by Eddington, and the currently-predicted value has since been validated to high accuracy.

Most accurate deflection results from radio astronomy

Radio astronomy, which uses radio waves instead of visible light, provides much more accurate results than the deflection of starlight observed by optical telescopes. Each October the Sun moves across the image of the quasar labeled 3C279 seen from Earth. Radio astronomers use this so-called **occultation** to measure the change in direction of the signal as—from our viewpoint on Earth—the source approaches the Sun, crosses the edge of the Sun, and moves behind the Sun. They employ an experimental technique called **very long baseline interferometry** (VLBI) that effectively uses two or more widely separated antennas as if they were a single antenna. This wide separation substantially increases the accuracy of observation. VLBI observations by E. Fomalont and collaborators measure a gravitational deflection to be a factor 1.9998 ± 0.0006 times the Newtonian prediction, in agreement with general relativity's prediction of 2 times the Newtonian result.

Since 1919, the gravitational deflection of light has become a powerful observational tool, as described in the remainder of this chapter.

PUTTING EINSTEIN TO THE TEST

No matter how revolutionary it was, no matter how beautiful its structure, our guide had to be experimentation. Equipped with new measuring tools provided by the technological revolution of the last twenty-five years, we put Einstein's theory to the test. What we found was that it bent and delayed light just right, it advanced Mercury's perihelion at just the observed rate, it made the Earth and the Moon fall the same [toward the Sun], and it caused a binary system to lose energy to gravitational waves at precisely the right rate. What I find truly amazing is that this theory of general relativity, invented almost out of pure thought, guided only by the principle of equivalence and by Einstein's imagination, not by need to account for experimental data, turned out in the end to be so right.

—Clifford M. Will

Galaxies not spherically symmetric.

Galaxies are not generally spherically symmetric, and light that passes through or close to a galaxy is *not* deflected according to Einstein's result (18). Light that passes *far* from a galaxy is deflected by an amount that can be *approximated* by Einstein's equation.

13.6 ■ GRAVITATIONAL MIRAGES

Derive and apply the gravitational lens equation.

Many rays can form an image.

So far this chapter has analyzed the deflection of a *single* ray of light by a point mass or a spherically symmetric center of attraction. But our view of the

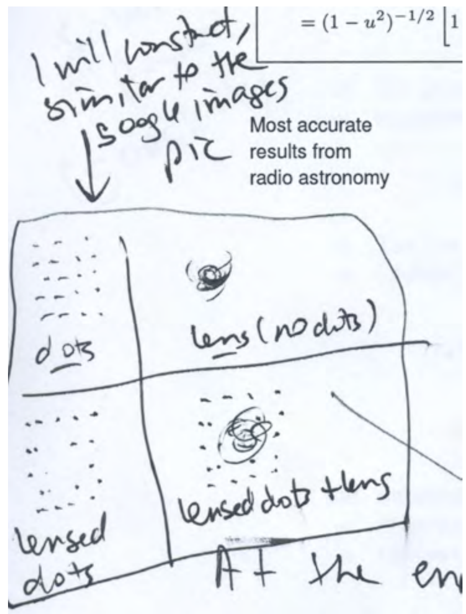


FIGURE 8 Prediction: A square array of distant light sources (upper left panel) is imaged by a galaxy (upper right panel) acting as a gravitational lens. The result is a distorted image of the square array (lower left panel). The lower right panel superposes gravitational lens and distorted image of the square array. [EB will provide final figure.]

heavens is composed of *many* rays. Many rays from a single source—taken together—can form an image of that source. We now examine gravitational lensing, the imaging properties of a spherically symmetric center of attraction. We already know that a gravitational lens differs radically from a conventional focusing lens (Figure 3); this difference leads to a distorted image we call a *gravitational mirage*. Figure 8 previews the image of a square array of distant sources produced by a galaxy that lies between those sources and us.

Gravitational image: *always* distorted.

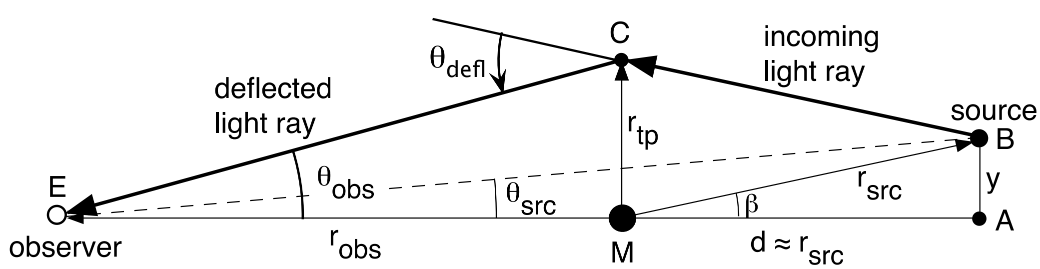


FIGURE 9 Construction for derivation of the gravitational lens equation in Box 2. View is far from a small deflecting mass M, so Euclidean geometry is valid. We assume that all angles are small and light deflection takes place at the turning point, $r_{\rm tp} \gg M$. Not to scale.

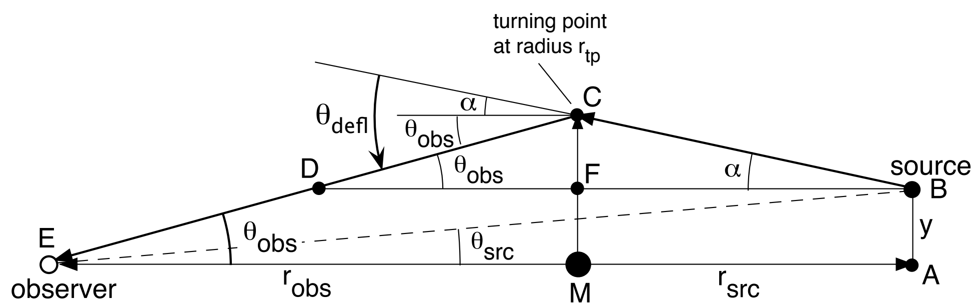


FIGURE 10 Added construction for derivation of the gravitational lens equation in Box 2. Not to scale.

Box 2. Gravitational Lens Equation

Here we derive the lens equation for a ray passing outside a spherically symmetric center of attraction of mass M. We put into this equation the angle $\theta_{\rm obs}$ at which the observer sees the source in the presence of an intermediate gravitational lens and it tells us the angle $\theta_{\rm src}$ at which the observer would see the source in the absence of that lens. The equation also contains the radial coordinate separations $r_{\rm obs}$ and $r_{\rm src}$ of observer and source from the lens.

Use the notation and construction lines in Figures 9 and 10. Assume that spacetime is flat enough so that we can use Euclidean space geometry (except in the immediate vicinity of the point mass M), assume that deflection takes place at the single turning point $r_{\rm tp}$, and finally assume that all angles are extremely small. By "extremely small," we mean, for instance, that in Figure 9, $\beta \approx y/r_{\rm src} \ll 1$. This means that the Euclidean distance d has the value

$$d = r_{\rm src}\cos\beta \approx r_{\rm src}(1-\beta^2/2) \approx r_{\rm src} \qquad (24)$$

to first order in β. Similarly, we can approximate $\sin\theta_{\rm src} \approx \theta_{\rm src}$ and $\sin\theta_{\rm obs} \approx \theta_{\rm obs}$. From triangle ABE in Figure 9

$$y = \theta_{\rm src}\left(r_{\rm obs} + r_{\rm src}\right) \qquad (25)$$

From triangle CEM in that figure:

$$r_{\rm tp} = \theta_{\rm obs} r_{\rm obs} \qquad (26)$$

In both Figures 9 and 10, the radial separation between C and M is $r_{\rm tp}$. In Figure 10, define $\overline{CF}$ as the coordinate separation between C and F, and use (26):

$$y = r_{\rm tp} - \overline{CF} = \theta_{\rm obs} r_{\rm obs} - \alpha r_{\rm src} \qquad (27)$$

We need to find an expression for the angle α. From triangle BCD in Figure 10 and the angles to the left of point C:

$$\alpha = \theta_{\rm defl} - \theta_{\rm obs} \qquad (28)$$

Equate the two expressions for y in (25) and (27) and substitute for α from (28)

$$\theta_{\rm src}\left(r_{\rm src} + r_{\rm obs}\right) = \theta_{\rm obs} r_{\rm obs} + \left(\theta_{\rm obs} - \theta_{\rm defl}\right) r_{\rm src} \qquad (29)$$

From (18) and (26):

$$\theta_{\rm defl} = \frac{4M}{r_{\rm obs}\theta_{\rm obs}} \qquad (30)$$

Substitute (30) into (29) and solve for $\theta_{\rm src}$:

$$\theta_{\rm src} = \theta_{\rm obs} - \frac{4Mr_{\rm src}}{\theta_{\rm obs} r_{\rm obs}(r_{\rm src} + r_{\rm obs})} \quad \text{(lens equation)} \qquad (31)$$

This is called the **lens equation** for a point mass M. The lens equation takes a simple form when expressed in terms of the Einstein angle $\theta_{\rm E}$ defined in equation (33):

$$\theta_{\rm src} = \theta_{\rm obs} - \frac{\theta_{\rm E}^2}{\theta_{\rm obs}} \quad \text{(lens equation)} \qquad (32)$$

Gravitational lens equation

In geometrical optics, the **lens equation** predicts the path of every ray that passes through a lens. For a spherically symmetric center of attraction—note this restriction!—every single ray that forms a gravitational image obeys Einstein's simple deflection equation (18). The present section uses this result to derive the **gravitational lens equation**. We assume that the observer is in flat interstellar space far from the lensing structure, so his frame is inertial. (An observer on Earth is sufficiently inertial for this purpose; Earth's atmosphere distorts incoming starlight far more than does Earth's gravitational deflection of light.)

The **source**

In practice, the source of light may be any radiant object, including all or part of a galaxy. So instead of "star" or "galaxy," we simply call this emitting object the **source** (subscript: **src**). The purpose of the gravitational lens equation is to find the unknown (not measured) angle $\theta_{\rm src}$ from the angle $\theta_{\rm obs}$ at which the observer sees the source. Box 2 carries out this derivation.

Einstein ring

When the source is exactly behind the imaging center of attraction—in other words when $y = 0$ and $\theta_{\rm src} = 0$ in Figures 9 and 10—then the deflection is identical on all sides of the lens, so the observer's image of the source is a ring, called the **Einstein ring** (lower left panel in Figure 4, Figures 11 and 12). In this case the observation angle $\theta_{\rm obs}$ takes the name **Einstein ring angle**, whose square is:

$$\theta_{\rm E}^2 \equiv \frac{4Mr_{\rm src}}{r_{\rm obs}(r_{\rm src}+r_{\rm obs})} \qquad \text{(Einstein ring angle)} \qquad (33)$$

$$\equiv \frac{4GM_{\rm kg}r_{\rm src}}{r_{\rm obs}c^2(r_{\rm src}+r_{\rm obs})} \qquad \text{(Einstein ring angle, conventional units)} \qquad (34)$$

where equation (34) employs conventional units. Definition (33) simplifies expression (31) for $\theta_{\rm src}$ in Box 2, leading to (32).

?

Objection 1. *Whoa! The caption to Figure 12 claims that the farthest source galaxy lies 22 billion light years distant. How can this be? Isn't the Universe less than 14 billion years old?*

!

Early in this book we emphasized that global coordinate separations have no dependable relation to measured quantities. Recall Section 2.7 and the

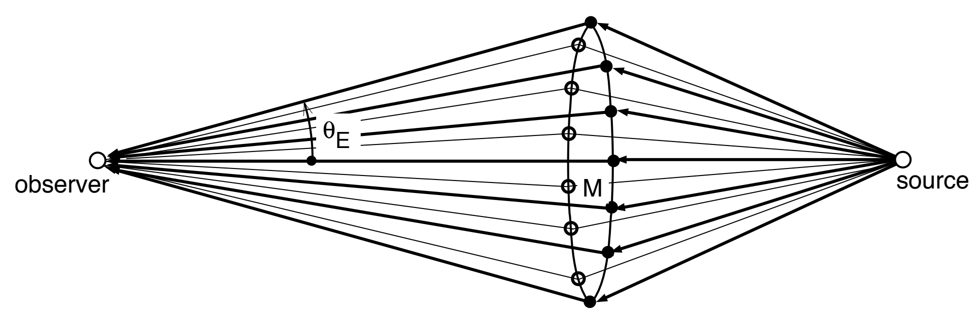


FIGURE 11 When the source, lens, and observer line up, then the deflection angle is the same on all sides of the lens, leading to the *Einstein ring*, whose observation angle $\theta_{\rm E}$ is given by equations (33) and (34). Note that fat rays deflect from one edge of the lens (solid dots) and narrow rays from the other side (little open circles). Not to scale.

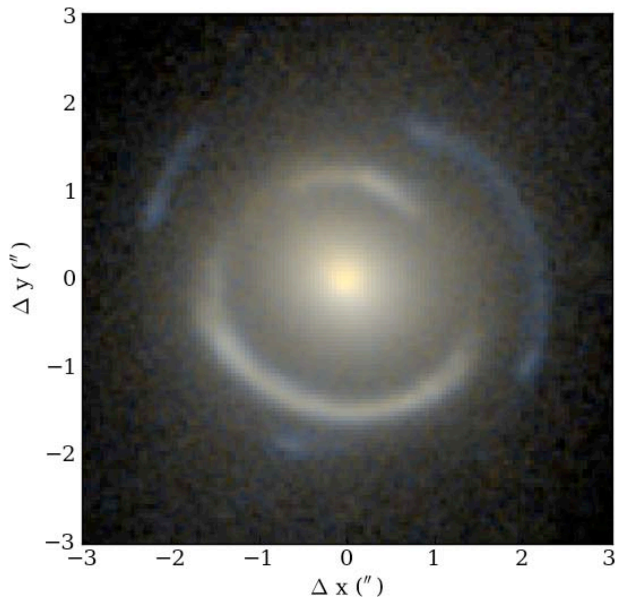


FIGURE 12 Two concentric Einstein rings that arise from two distant galaxy sources directly behind a foreground massive galaxy lens labeled SDSS J0946+1006. The horizontal and vertical scales are marked in units of 1" = one arcsecond. Redshifts of light from the three galaxies lead to the following estimates of their model distances from Earth: lens galaxy, 3.0 billion light years; nearest source galaxy, 7.4 billion light years; farthest source galaxy, 22 billion light years.

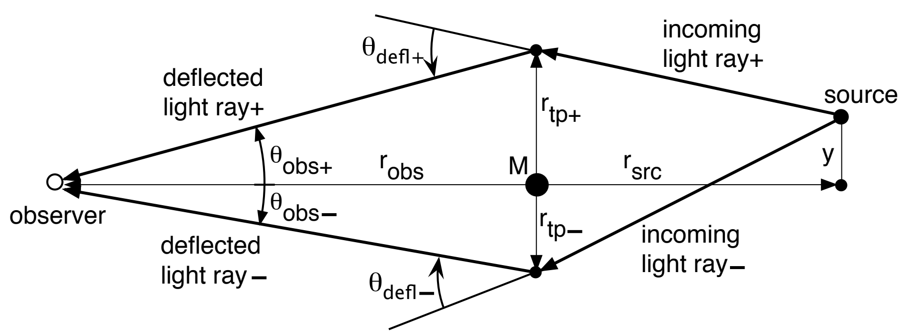


FIGURE 13 Two images of the star from (36). Not to scale.

First Strong Advice for this Entire Book (Section 5.6): "To be safe, it is best to assume that global coordinates never have any measurable meaning. Use global coordinates only with the metric in hand to convert a mapmaker's fantasy into a surveyor's reality." Chapter 15 shows how a cosmic metric translates coordinate differences into observable quantities like redshift—and also into estimated "distances" that depend on a model, such as those quoted in the captions of Figures 12 and 14.

QUERY 3. Use Einstein ring angle to measure the mass of a lensing galaxy.

A. Use the horizontal or vertical axis label in Figure 12 to make yourself a ruler in units of arcseconds. With this ruler measure the average angular radii of the two Einstein rings. From these average radii and the source and lens model distances given in the caption, calculate two independent estimates of the mass of the lensing galaxy in units of the mass of our Sun. Do the results agree to one significant figure?

B. Why does the lens equation (18) work for these Einstein rings, despite the warnings at the end of Box 1 and in the final paragraph of Section 13.5 that Einstein's deflection equation works only for a spherically symmetric lens? Does the image of the lensing structure in the center of Figure 12 give you a hint?

C. Figure 12 demonstrates that the Einstein ring angle $\theta_{\rm E}$ created by a galaxy lens is observable with modern technology, as Zwicky predicted in 1937 (Section 13.1). In contrast, the Einstein ring angle for a star lens is too small to observe, as Einstein predicted in 1936. To demonstrate this, calculate the Einstein ring angle lensed by a star with the mass of our Sun located at the center of our galaxy 26 000 light years distant, with the source twice as far away. Order of magnitude result: $\theta_{\rm E} \sim 10^{-3}$ arcsecond. Give your answer to one significant digit. [My answer: 7.2×10^{-4}.] Would you expect to see this image as an Einstein ring, given the resolution of Figure 12?

Equation (32) leads to the following quadratic equation in $\theta_{\rm obs}$:

$$\theta_{\rm obs}^2 - \theta_{\rm src}\theta_{\rm obs} - \theta_{\rm E}^2 = 0 \qquad (35)$$

Two images

This equation has two solutions which correspond to two images of the source in the x, y plane of Figure 13:

$$\theta_{\rm obs\pm} = \frac{\theta_{\rm src}}{2} \pm \frac{1}{2}\left(\theta_{\rm src}^2 + 4\theta_{\rm E}^2\right)^{1/2} \qquad (36)$$

QUERY 4. Use double images to measure lensing galaxy mass. Estimate of the mass of the gravitational lens in Figure 14, as follows:

A. Measure the angular separation $|\theta_{\rm obs+} - \theta_{\rm obs-}|$ of the two images in arcseconds.

B. Measure the angular separation $|\theta_{\rm obs-}|$ of the lensing galaxy and the quasar image closest to it in arcseconds.

C. Use equation (36) to determine the separate values of $\theta_{\rm src}$ and $\theta_{\rm E}$.

D. From your value of $\theta_{\rm E}$ and model distances given in the caption of Figure 14, deduce the mass of the lensing galaxy in units of solar mass to one significant figure. Compare this result with the mass of the lensing galaxy in the system SDSS J0946+1006 of Figure 12 that you calculated in Query 3.
Note: The mass found here is only approximate, because the lens is not spherically symmetric. More complex modeling finds a lens mass $(3.9 \pm 1.2) \times 10^{14} M_{\rm Sun}$ (Kundić et al in the references).

Comment 2. Local time delay between images

Light from the separate images in Figure 14 travel along different paths in global coordinates between source and observer, as shown in Figure 13. If the intensity of the source changes, that change reaches the observer with different local time delays in the two images. Kundić et al (see the references) measured the local time delay between the images in Figure 14 and found it to be 417 ± 3 days: more than one Earth-year. This difference in locally-measured time delay seems large, but is a tiny fraction of the total lapse of global t along either path from that distant source to us.

Gravitational lensing detects dark matter.

Gravitational lensing by galaxies provides some of the strongest evidence for the existence and importance of **dark matter**. Galaxy masses obtained by gravitational lensing are much larger than the combined mass of all the visible stars in the measured galaxies. Most of the *matter* in the Universe is not atoms but a mysterious form called dark matter. Chapter 15 discusses the cosmological implications of this result. That chapter also examines, in

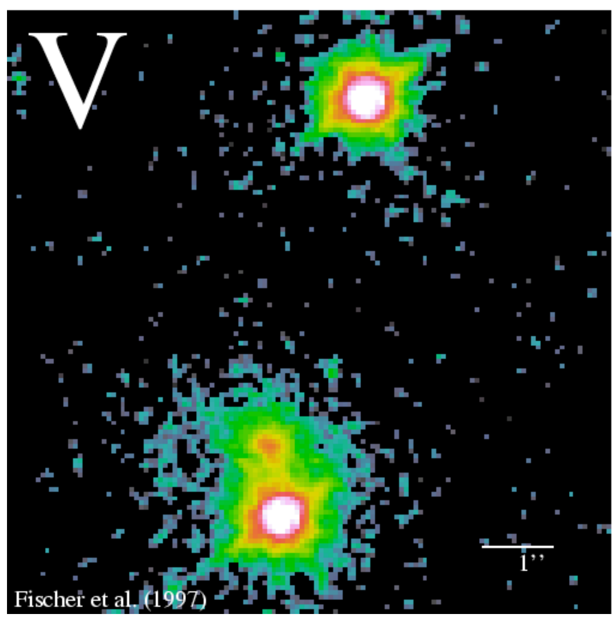


FIGURE 14 Double image from microwave observations of the distant quasar imaged by a foreground lensing structure—Figure 13 and equation (36). This is the first gravitationally lensed object, called QSO 0957+561A/B, observed in 1979 by Walsh, Carswell, and Weymann. The small patch above the lower image is the lensing galaxy. Redshifts of light from the distant quasar and the lensing galaxy lead to the following estimates of their model distances from Earth: lens galaxy, 4.6 billion light years; quasar source, 14.0 billion light years. The two images of the distant quasar are not collinear with that of the lens, which demonstrates that the lens is not spherically symmetric, as our analysis assumes.

addition to matter, the presence and importance of the mass-equivalent of light, neutrinos, and a larger and still more mysterious contribution called **dark energy**.

13.7 ■ MICROLENSING

The image brightens, then dims again.

Microlensing: when images cannot be resolved

Suppose that our detector—telescope, microwave dish, X-ray imaging satellite, or some other—cannot resolve the separate images of a distant star caused by an intermediate gravitational lens. Einstein warned us about this in 1936 (Section 13.1). In this case we see only one image of the source. Nevertheless, the intermediate lens directs more light into our telescope than would otherwise arrive from the distant source. We call this increase of light **microlensing**. How can we use that increased amount of light to learn about the lensing structure that lies between the source and us? We begin with a necessary set of definitions.

Instead, detect increased brightness of the single image.

DEFINITION 1. Solid angle

To measure star patterns in the night sky—whether detected by visible light, microwaves, infrared, ultraviolet, X-rays, or gamma rays—astronomers record the *angle* between any given pair of images. *For astronomers, angle is the only dependable geometric measure of the heavens.* The cross-hatched region of a distant source in Figure 15 has a length and width both measured in angle. We call the resulting measure **solid angle**. Solid angle is angular area, measured in square arcseconds or square radians (square radians has the technical name **steradians**). In the following we derive the *ratio* of solid angles, defined as *magnification*.

Solid angle

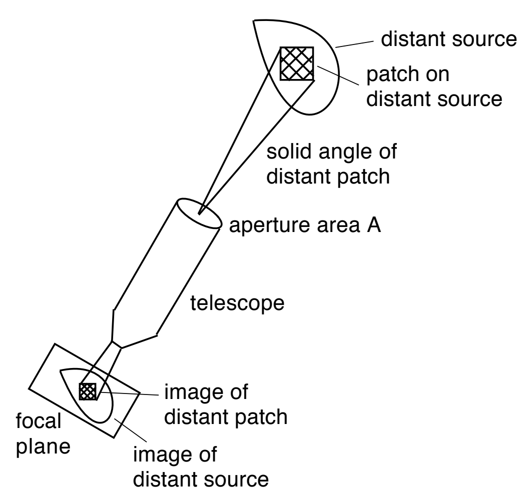


FIGURE 15 Figure for Definition 2 of Intensity, Flux, and Magnification.

DEFINITION 2. Intensity, Flux, and Magnification

Figure 15 helps to define intensity, flux, and magnification. We impose two conditions on these definitions: (1) The definitions must describe light from the source and not the instrument we use to measure it. (2) The distant object being observed is not a point source, so we can speak of a patch of solid angle on that source.

A camera attached to a telescope of aperture area A (Figure 15) displays the image of a patch with a given solid angle on the sky, say the portion of a distant galaxy. The intensity I of the light is defined as:

Intensity

$$I \equiv \frac{\text{total energy of light from patch recorded by camera}}{\text{local time} \times \text{aperture } A \times \text{solid angle of patch in the sky}} \quad (37)$$

The flux F of the source is the total energy striking the camera plane from the entire source per unit local time and per aperture area A :

Flux

$$\text{Flux} \equiv F = \int_{\text{over source}} (\text{Intensity}) d(\text{solid angle of source}) \quad (38)$$

Now place a gravitational lens between source and detector. The image in the focal plane will be changed in size and also distorted. However, its intensity is not changed. *Example for a conventional lens:* Hold a magnifying lens over a newspaper. The lens directs more light into your eye; the flux increases. However, the area of the newspaper image on your retina increases by the same ratio. *Result:* The newspaper does not look brighter; you see its intensity as the same. However, a larger image with the same intensity means more flux, more energy, in the same ratio as solid angles, namely magnification. So magnification is equal to the ratio of fluxes, even when your detector cannot resolve the larger image. The result for a magnifying lens is also the result for a gravitational lens: the flux increases in proportion to the magnification.

Magnification

$$\text{Magnification} \equiv \frac{(\text{solid angle with gravitational lens})}{(\text{solid angle without gravitational lens})} \quad (39)$$

$$= \frac{F(\text{with gravitational lens})}{F(\text{without gravitational lens})} \quad (40)$$

Figure 16 and Box 3 derive the magnification of a point gravitational lens.

Use microlensing to detect and study invisible lensing structure.

Astronomers use microlensing to detect the presence and estimate the mass of an intermediate lensing object, for example a star that is too dim for us to see directly.

?

Objection 2. *Wait a minute! When we see a distant source, it is just a source like any other. How can we tell whether or not the flux from this*

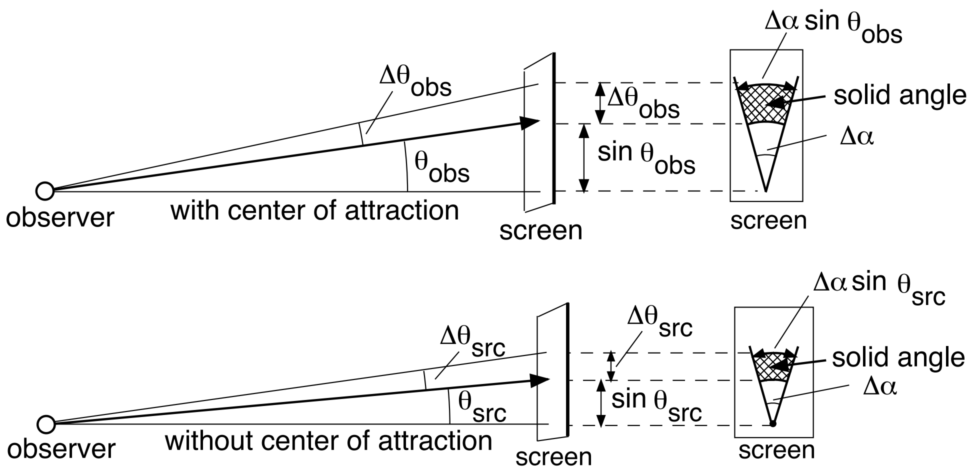


FIGURE 16 Magnification of the image of an extended source by a center of attraction acting as a gravitational lens. A patch on the source is the cross-hatched solid angle at the right of the lower panel; the corresponding patch on the image is the cross-hatched solid angle at the right of the upper panel. The magnification is the ratio of the upper to the lower cross-hatched solid angles. Box 3 employs this figure to derive the magnification of a gravitational lens.

Box 3. Image Magnification

Figure 16 shows cross-hatched solid angles whose ratio defines the magnification of an extended source by a gravitational lens. Magnification is defined as the ratio of the cross-hatched solid angle patch in the upper panel (with the center of attraction present) to the cross-hatched solid angle patch in the lower panel (with no center of attraction present).

To find this ratio, pick a wedge of small angle $\Delta\alpha$, the same for both panels. The radius of the cross-hatched solid angle for each wedge is proportional to $\sin\theta_{\rm obs}$ in the upper panel and $\sin\theta_{\rm src}$ in the lower panel. The angular spread in each case is $\Delta\alpha$ times the sine factor.

Then the magnification, equal to the ratio of solid angles with and without the center of attraction, becomes:

$$\text{Mag} = \left|\frac{\sin\theta_{\rm obs}\Delta\theta_{\rm obs}}{\sin\theta_{\rm src}\Delta\theta_{\rm src}}\right| \approx \left|\frac{\theta_{\rm obs}d\theta_{\rm obs}}{\theta_{\rm src}d\theta_{\rm src}}\right| \qquad (41)$$

where we add absolute magnitude signs to ensure that the ratio of solid angles is positive. In the last step of (41) we assume that observation angles are small, so that $\sin\theta \approx \theta$ and $\Delta\theta \approx d\theta$. Now into (41) substitute $d\theta_{\rm src}$ from the differential of both sides of (32), with $\theta_{\rm E}$ a constant:

$$d\theta_{\rm src} = d\theta_{\rm obs} + \frac{\theta_{\rm E}^2}{\theta_{\rm obs}^2}d\theta_{\rm obs} \qquad (42)$$

Equation (41) becomes

$$\begin{aligned}\text{Mag} &= \left|\frac{\theta_{\rm obs}d\theta_{\rm obs}}{\left(\theta_{\rm obs} - \dfrac{\theta_{\rm E}^2}{\theta_{\rm obs}}\right)\left(1 + \dfrac{\theta_{\rm E}^2}{\theta_{\rm obs}^2}\right)d\theta_{\rm obs}}\right| \\ &= \left|\left(1 - \frac{\theta_{\rm E}^2}{\theta_{\rm obs}^2}\right)^{-1}\left(1 + \frac{\theta_{\rm E}^2}{\theta_{\rm obs}^2}\right)^{-1}\right| \qquad (43)\end{aligned}$$

So finally,

$$\text{Mag} = \left|1 - \frac{\theta_{\rm E}^4}{\theta_{\rm obs}^4}\right|^{-1} \qquad (44)$$

source has already been increased by an intermediate gravitational lens? For example, it might just be a brighter star.

Good point. For a static image—one that does not change as we watch it—we cannot tell whether or not an intermediate lens has already changed the flux. However, if the source and lensing object move with respect to one another—which is the usual case—then the total flux changes with local time, growing to a maximum as source and gravitational lens line up with one another, then decreasing as this alignment passes. Figure 17 displays a theoretical family of such curves and Figure 18 shows the result of an observation.

Figure 13 shows that the observer receives two images of the source. Even though we cannot currently resolve these two images in microlensing, the total flux received is proportional to the summed magnification of both images:

$$\text{Mag}_{\text{total}} = \text{Mag}\left(\theta_{\text{obs}+}\right) + \text{Mag}\left(\theta_{\text{obs}-}\right) \equiv \text{Mag}_+ + \text{Mag}_- \tag{45}$$

Total magnification equals increased flux.

where (36) gives $\theta_{\text{obs}\pm}$.

Now we descend into an algebra orgy: Divide both sides of (36) by θ_{E} and substitute $q \equiv \theta_{\text{src}}/\theta_{\text{E}}$. Insert the results into (44). The expression for the separate magnifications M_+ and M_- of the two images become:

$$\text{Mag}_\pm = \left| \frac{1 + 2q^2 + \dfrac{q^4}{2} \pm \dfrac{q}{2}\left(q^2+2\right)\left(q^2+4\right)^{1/2}}{2q^2 + \dfrac{q^4}{2} \pm \dfrac{q}{2}\left(q^2+2\right)\left(q^2+4\right)^{1/2}} \right| \qquad \text{where} \quad q = \frac{\theta_{\text{src}}}{\theta_{\text{E}}} \tag{46}$$

Substitute this result into (45) to find the expression for total magnification.

$$\text{Mag}_{\text{total}} = \frac{q^2+2}{q\left(q^2+4\right)^{1/2}} \qquad \text{where} \quad q = \frac{\theta_{\text{src}}}{\theta_{\text{E}}} \tag{47}$$

Comment 3. Variation of $\text{Mag}_{\text{total}}$ with q

It may not be obvious that smaller q results in larger total magnification. Convince yourself of this by taking the derivative of (47) with respect to q or by plotting its right hand side.

The maximum magnification—the maximum brightness of the microlensed background source—occurs for the minimum value of q:

$$\text{Mag}_{\text{total,max}} = \frac{q_{\text{min}}^2+2}{q_{\text{min}}\left(q_{\text{min}}^2+4\right)^{1/2}} \qquad \text{where} \quad q_{\text{min}} = \frac{\theta_{\text{src,min}}}{\theta_{\text{E}}} \tag{48}$$

What does the observer see as a the source passes behind the lens? To answer this question, give the source angle a time derivative in equation (47):

$$q = \frac{\dot{\theta}_{\text{src}}}{\theta_{\text{E}}}\left(t - t_0\right) \tag{49}$$

Proper motion

The symbol $\dot{\theta}_{\text{src}}$ is the angular velocity of the moving source seen by the observer—called its **proper motion** by astronomers—and t_0 is the observed

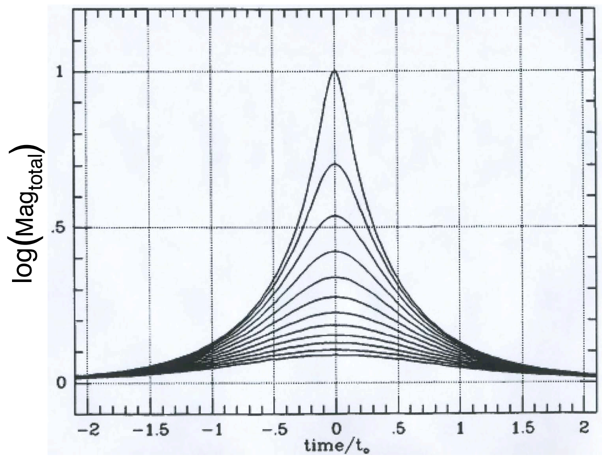


FIGURE 17 Log of total magnification $\mathrm{Mag}_{\mathrm{total}}$ (vertical axis) due to microlensing as the source moves past the deflecting mass, with t_0 a normalizing local time of minimum separation. Different curves, from top to bottom, are for the 12 values $q_{\mathrm{min}} = 0.1, 0.2 \ldots 1.1, 1.2$ in equation (47), respectively. From a paper by Paczynski, see the references.

time at which the source is at the minimum angular separation $\theta_{\mathrm{src,min}}$. Substitute (49) into the definition of q in (47) to predict the local time dependence of the apparent brightness of the star. Figure 17 shows the resulting predicted set of light curves for different values of $q_{\mathrm{min}} = \theta_{\mathrm{src,min}}/\theta_{\mathrm{E}}$.

“Achromatic” gravitational lens

Comment 4. Gravitational lenses are achromatic.
Equations of motion for light around black holes are exactly the same for light of every wavelength. Technical term for any lens with this property: **achromatic**. Gravitational lenses often distort images terribly, but they do not change the color of the source, even when “color” refers to microwaves or gamma rays. The achromatic nature of a gravitational lens can be important when an observer tries to distinguish between increased light from a source due to microlensing and increased light due to the source itself changing brightness. A star, for example, can increase its light output as a result of a variety of internal processes, which most often changes its spectrum in some way. In contrast, the increased flux of light from the star due to microlensing does not change the spectrum of that light. Therefore any observed change in flux of a source without change in its spectrum is one piece of evidence that the source is being microlensed.

Predicted microlensing curve observed

Figure 18 shows a microlensing curve for an event labeled OGLE 2005-BLG-390. The shape of the observed magnification curve closely follows the predicted curves of Figure 17 when converted to a linear scale.

What can we learn from an observation such as that reported in Figure 18? In Query 5 you explore two examples.

QUERY 5. Results from Figure 18

A. From the value of the magnification in Figure 18, find the value of q_{min}.

B. Measure the observed time between half-maximum magnifications in Figure 18; call this $2(t_{1/2} - t_0)$. The horizontal axis in Figure 17 expresses this observed time as a function of $\dot{\theta}(t - t_0)/\theta_{\rm src,min}$. (Careful: this is a semi-log plot.) From these two results calculate the value of the quantity $\dot{\theta}/\theta_{\rm src,min}$.

Exoplanet detected

The tiny spike on the right side of the curve in Figure 18—magnified in the inset labeled "planetary deviation"—shows another major use of microlensing: to detect a planet orbiting the lensing object. The term for a planet around a star other than our Sun is **extra-solar planet** or **exoplanet**.

How exoplanet detection is possible.

The presence of a short-duration, high-magnification achromatic spike in a long microlensing event is evidence for an exoplanet, which causes additional deflection and magnification of one of the two images. This additional magnification results from the small value of q in equation (48) and has much shorter duration $\theta_{\rm src,min}/\dot{\theta}_{\rm src}$ than that due to the primary lens star because the minimum separation between the exoplanet and light ray is very small, as shown in the middle panel of Figure 19.

Comment 5. Shape of the exoplanet curve

The faint background curve in the "Planetary deviation" inset of Figure 18 has the shape similar to curves in Figure 17 predicted for a static point-mass planet.

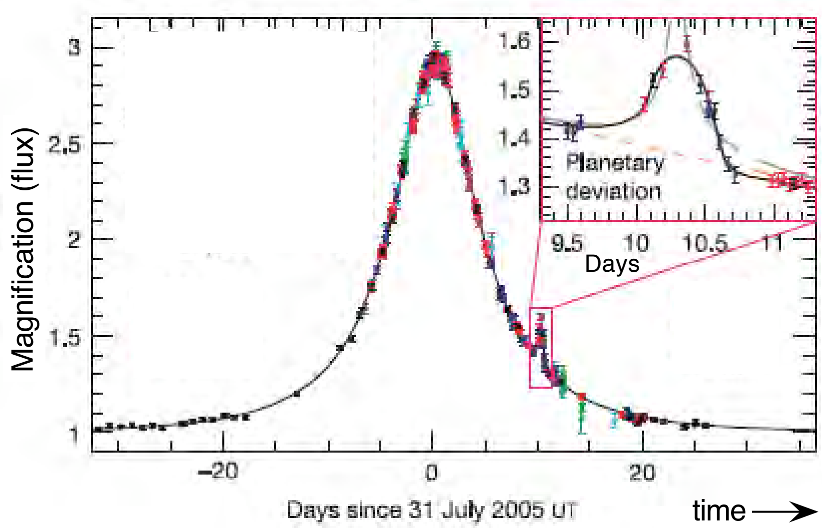


FIGURE 18 Microlensing image with the code name OGLE 2005-BLG-390. The lens is a dwarf star, a small relatively cool star of approximately 0.2 solar mass. Observed time along the horizontal axis shows that the variation of intensity can take place over days or weeks. The abbreviation UT in the horizontal axis label means "universal time," which allows astronomical measurements to be coordinated, whatever the local time zone of the observer. The inset labeled "planetary deviation" detects a planet of approximately 5.5 Earth masses orbiting the lens star.

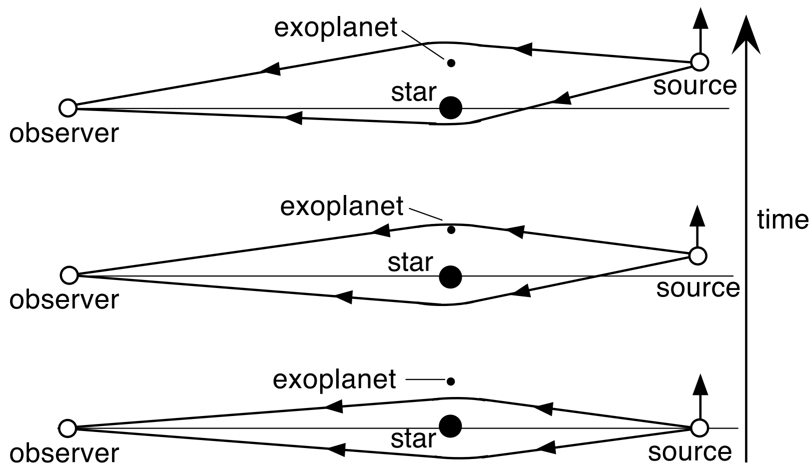


FIGURE 19 Schematic diagram of the passage of the source behind the lensing star with an exoplanet that leads to the small spike in the local time-dependent flux diagram of Figure 18. Observer time increases from bottom to top. The bottom panel displays the alignment at local time $t - t_0 = 0$ (Figure 17), when the source is directly behind the lens, which results in maximum flux from the source at the observer. The middle panel shows the alignment that leads to the maximum of the little spike in Figure 18. Figure not to scale.

Question: Why does the *shape* of exoplanet-induced magnification curve differ from this prediction (as hinted by the phrase “planetary deviation”)? *Answer:* Because the planet moves slightly around its mother star during the microlensing event, so $\dot{\theta}$ is not constant.

Analysis of the exoplanet spike on a microlensing flux curve is just one of several methods used to detect exoplanets; we do not describe other methods here.

13.8 ■ REFERENCES

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