

The Simple Harmonic Oscillator

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BOUND STATES -- AT LAST!

Most of the electrons around us are bound up in atoms and molecule—and thank goodness. Loose electrons are dangerous to life. So is lack of electrons. Are you taking antioxidant vitamins to keep molecules with too few electrons from clogging your arteries?

But what ARE bound states of electrons? And why are these bound states so STABLE, so unchanging with time? Almost every object around us maintains its shape, hardness, and dependability over time—again, thank goodness.

And what have bound states to do with electrons exploring paths through spacetime? How are bound states predicted by the propagators that summarize the results of the electron exploring ALL worldlines between initial and final events in a given potential?

One of the simplest binding potentials is that of the so-called *simple harmonic oscillator*, or SHO for short. The SHO potential is parabolic in shape; the value of the potential increases with the SQUARE of the distance from the center of attraction, creating a kind of “bucket” or “cup” that contains or binds the charged particle. See Figure 1.

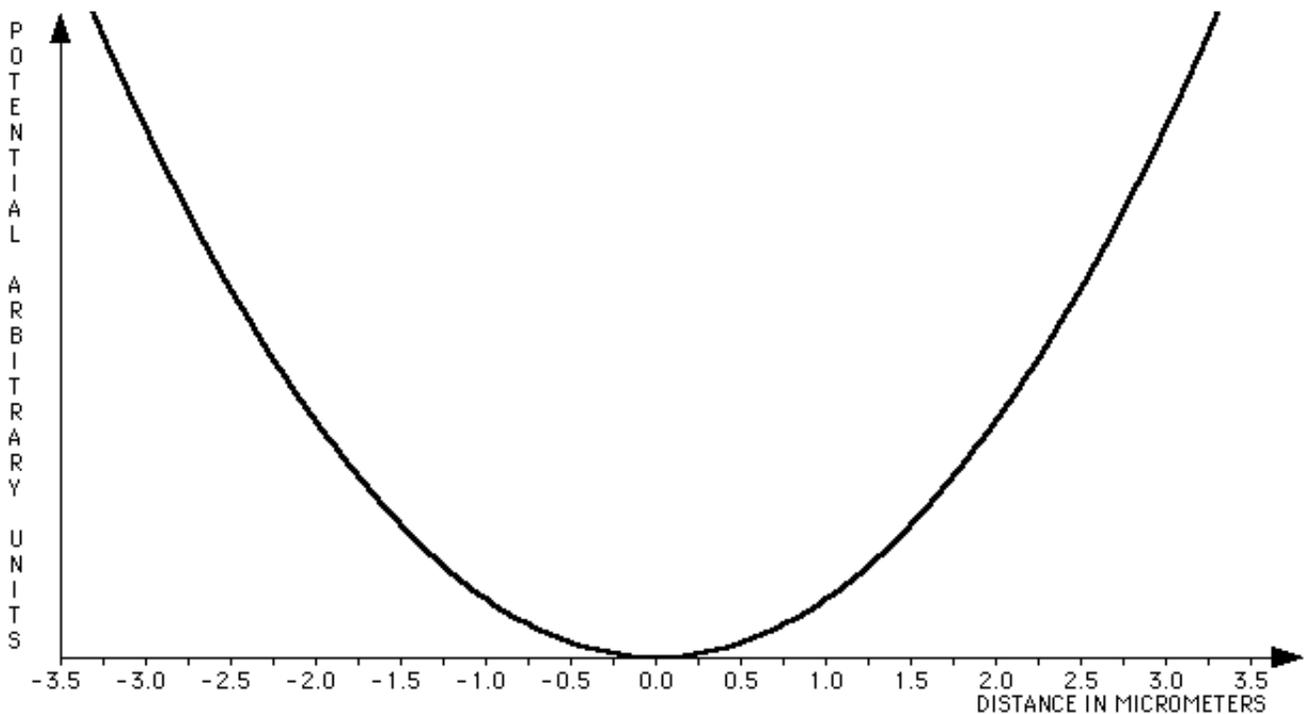


Figure 1. The simple harmonic oscillator (SHO) potential. The potential energy varies with the square of the distance from the center. Such a potential can be used to bind a charged particle.

In a previous unit, Worldlines for the Quantum Particle, we traced out worldlines in a binding potential. By varying the profile of each worldline, we discovered the shape corresponding to the minimum number of rotations of the quantum stopwatch between fixed events of emission and detection. A particle following a minimum-rotation worldline can start out moving away from the center of attraction but later curve back toward the center. This particle is *bound* in the confining potential. A minimum-rotation worldline traces out the motion of a high-mass particle in the classical limit.

But a quantum particle of small mass, whether free or in a binding potential, does not follow a single worldline or even a narrow pencil of worldlines. Rather it can be correctly described only as exploring ALL worldlines between events of emission and detection. The results of this all-worldline exploration can be duplicated using the *propagator* for the given particle in the given binding potential.

In this unit we explore a quantum particle bound in the SHO potential. You are not asked to find the propagator for the SHO—it is a bit complicated! Instead, you use a piece of software into which this propagator has already been programmed. Your task is to explore the consequences of this propagator for the time development of various initial wavefunctions in the SHO potential. The propagator is applied between *every* arrow in the initial wavefunction and *each* event in the final wavefunction in order to construct the resulting arrow at each final event.

One of the consequences of this analysis is of fundamental consequence for our Universe. There are some unique initial wavefunctions each of whose arrows rotate in unison with the passage of time but each of these arrows do not change length with time. One example is shown in Figure 2. The arrows all rotate in unison, but each keeps its same length.

Now, the probability of finding the particle at a given detection event is proportional to the square of the length of the arrow at that event. But arrow lengths do not change with time for this unique wavefunction. Therefore the probability does not change with time, as shown in Figure 3.

Because probability does not change with time, these unique wavefunctions are called **stationary**, and the quantum state which each such wavefunction describes is called a **stationary state**.

In Nature, almost every binding potential has stationary states. Each stationary state has a unique energy. At a low enough temperature, a bound system tends to drop into its stationary state of lowest energy.

A different binding potential (the Coulomb potential of the hydrogen atom, the simple harmonic oscillator potential) will have a different set of stationary states and energies. But each such stationary state is *stationary*: the probability of finding the bound particle at a given location does not change with time.

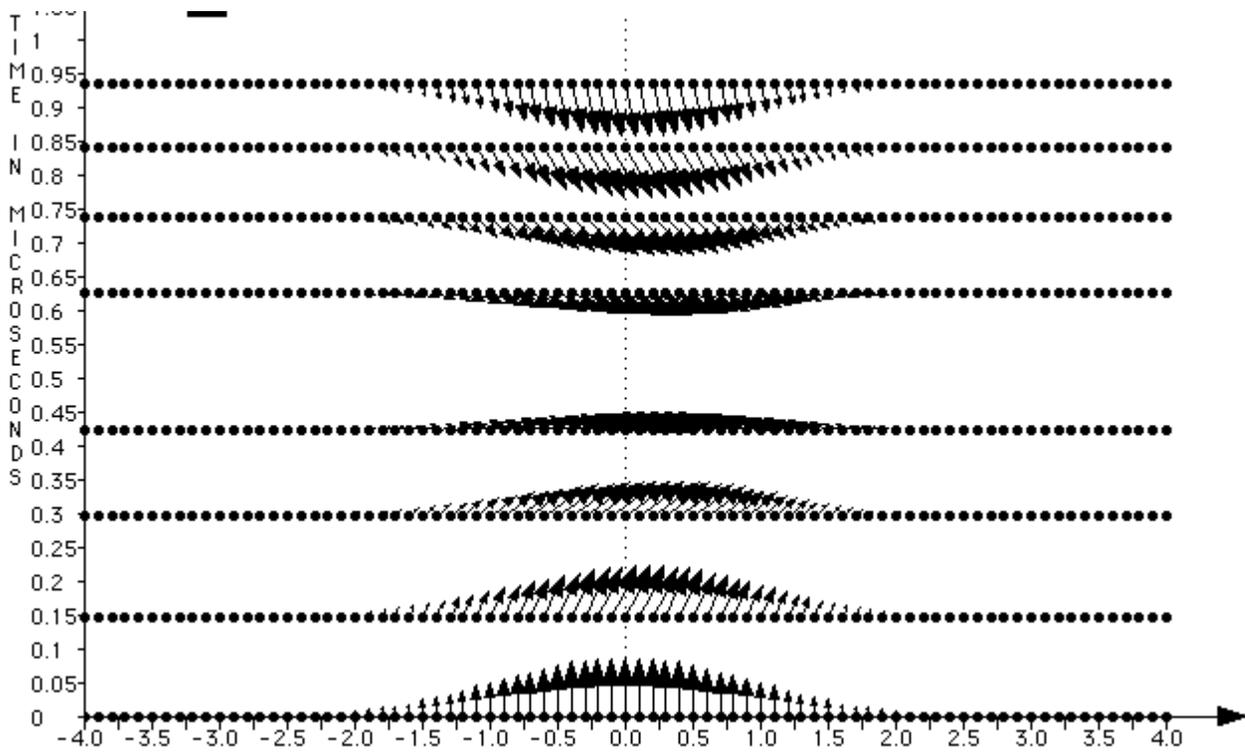


Figure 2. The lowest-energy stationary state for the simple harmonic oscillator. The arrows rotate with time but do not change length.

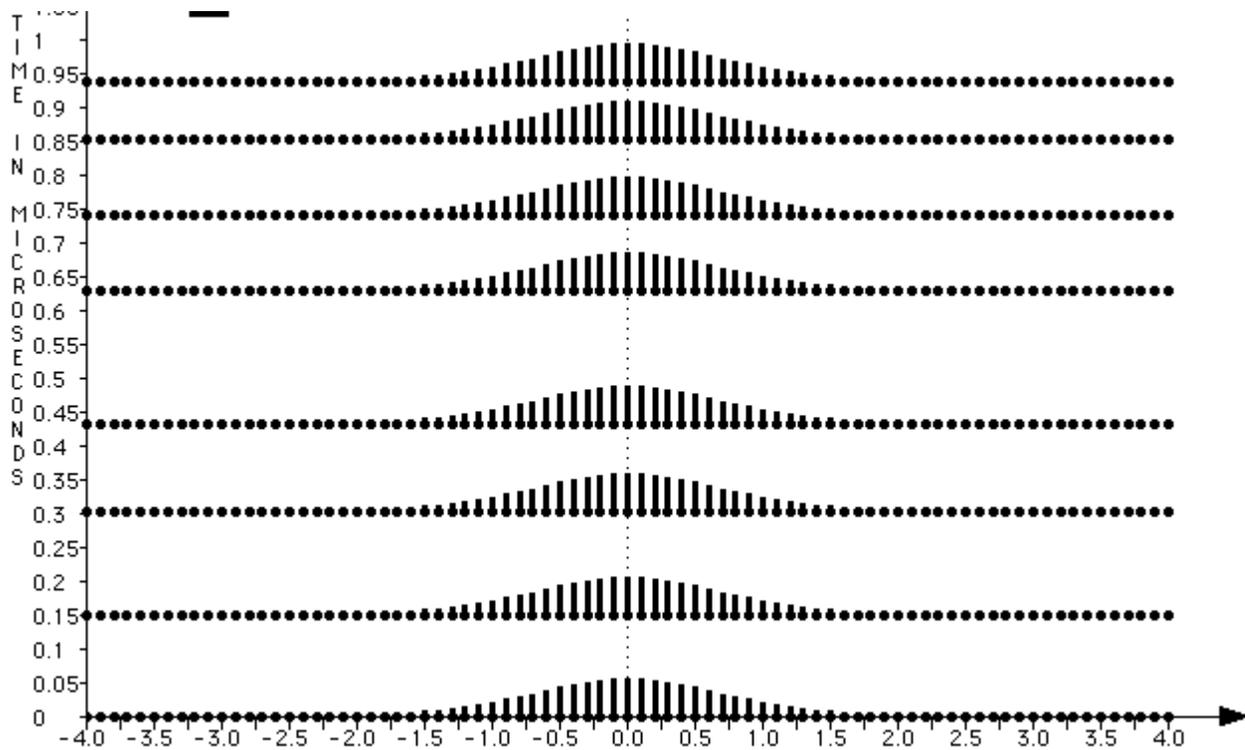


Figure 3. The probability function for the stationary state of Figure 2. The probability does not change with time. Thus this state earns the name “stationary.”

PART ONE: FOOLING AROUND WITH THE SHO SOFTWARE

Start out by fooling around with the SHO software. Be bold! Do NOT turn in results of this initial exploration, but do not neglect it either; it will inform your understanding and help in the exercises which follow. At this point in the course we are all researchers!

Here are some things to try:

SHO POTENTIAL

Start with the default SHO potential—the one present when you start up the program. Using the CHOOSE WAVEFUNCTION button, try all the wavefunctions, labeled with letters A thru K. How does each of these develop with time in the SHO potential?

Press CHANGE DISPLAY and try all three display modes: ARROWS and COLOR and PROBABILITY. How do the screen pictures differ for these different modes.

Wavefunctions labeled F, G, H, and I are called STATIONARY, that is unchanging with time. WHAT IS IT that does not change with time? The arrows? the colors? the probability?

How do the NON-stationary wavefunctions change with time in the SHO potential? Does each initial wavefunction—or any of them—re-assemble into its original form after a particular time?

What, if anything, is special about wavefunctions J and K, which are labeled SUPERPOSITION? Do they behave in ways different from the other initial wavefunctions?

ZERO POTENTIAL

Change to zero potential—so that the particle is free, moving without any potential it influence it. PREDICT what will happen as time passes to each of the initial wavefunctions A thru K in the zero potential. Check out your predictions.

Redo ALL the explorations described above for zero potential. Are the SHO “stationary” wavefunctions still “stationary” in zero potential?

PART TWO: EXERCISES

NOTE: For some choices of time, you will get an INVALID WAVEFUNCTION message. For an explanation of this message, use the Error menu after Restart. When it is important to know what happens at an “invalid” time you may need to “straddle” this time on either side and guess what actually happens AT that time.

Q1. How much time does it take for arrows in the SHO stationary state #0 (wavefunction F) to rotate ONE-HALF turn (from arrows straight up to arrows straight down)? Therefore, how much time will it take for the arrows in this state to rotate ONE FULL turn? Therefore, what is the FREQUENCY, call it f_0 , of this rotation, in megacycles/second (called megahertz)? (In Question Q1 you use one-half turn because one full turn takes longer than the extent of the vertical time scale.)

Q2. How much time does it take for arrows in the SHO stationary state #1 (wavefunction G) to rotate ONE-FULL turn? Therefore, what is the FREQUENCY, call it f_1 , of this rotation? Give your answer in megahertz.

Q3. Same question for SHO stationary state #2 (wavefunction H). Call the frequency f_2 .

Q4. Same question for SHO stationary state #3 (wavefunction I). Call the frequency f_3 .

Q5. What is the value of the CLASSICAL frequency, call it f_{cl} , for the default SHO potential? Derive the answer using the classical period T given in the lower left of the screen. What is the relation between f_{cl} and f_0 , the frequency of rotation of the arrows in the zeroth energy state?

Q6. ASSUME that the ENERGY E of a bound state is proportional to the frequency f with which its arrows rotate, according to Einstein's famous equation $E = hf$. Here h is Planck's constant. Use the results of questions Q1 through Q5 to PREDICT a general formula for the energies of the stationary states of the SHO in terms of:

(a) the symbol h for Planck's constant, and

(b) the symbol f_{cl} for the classical frequency, and

(c) the symbol n for the stationary state ($n = 0, 1, 2, 3, \dots$), numbered starting with zero as shown in the SHO program."

COMMENT: Stationary states for EVERY binding potential have arrows that rotate in unison. And these arrows ALWAYS rotate faster for higher energy states. However, the FORMULA for the energy as a function of state number n is different for different potential functions. Example: For the hydrogen atom the energy is proportional to MINUS $1/n^2$, where n is a positive integer that starts with the number one. For wavefunctions in three-dimensional binding potentials, there will usually be more than one quantum number, for example a number for angular momentum as well as for energy.

Q7. Using the software to help you to decide, state who is right in the following, Rachel or Andrew.

RACHEL says, "Each of the wavefunctions for the superposition states J and K returns to its original form after one period $T = 1/f_{cl}$ of the classical harmonic oscillator."

ANDREW says, "No, the PROBABILITY for each of the superposition states J and K returns to its initial form after one period, but the WAVEFUNCTION does not."

DISCUSSION: “SLOSHING STATES”

(Edited interchange with a student.)

PAUL ASKS:

One thing that puzzles me about the non-stationary states: Are they valid states for an electron bound in an atom? It seems to me that we could have an electron moving along in no potential (like wavefunctions C or D), which then encounters a potential (an atom), causing its wavefunction to evolve as we see in SHO. Those wavefunctions are not stationary, though they seem to stay bound around the zero position. So it seems like the electron can be in some state that isn't $n=(0, 1, 2, \text{etc.})$. Are they some superposition of the stationary states?

EDWIN REPLIES:

Paul, you are onto something important here. Think of water in a pail. It can slosh around apparently randomly. It can also vibrate sinusoidally in time with various different unique frequencies—each unique frequency having its unique wave pattern.

Same for an electron in the SHO potential (or in a single-electron atom). It can “slosh around” apparently randomly. Or it can be in a state whose arrows rotate in unison. These uniform-rotation states are the stationary states. The wavefunctions for stationary states are different for the SHO potential and the potential in an atom. However, for stationary states in either potential the PROBABILITY does not change at all with time.

And you are onto something even more profound: Every “sloshing” pattern of water in a pail can be constructed of a carefully-chosen superposition (adding up at every point) of the unique sinusoidally-varying patterns. And every “sloshing” wavefunction of an electron in the SHO potential or an atom can be constructed of a carefully-chosen superposition (adding up the arrows at every point) of stationary wavefunctions in that potential.

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