

The Propagator for a Free Particle

Edwin F. Taylor
July 2000

I. EXPLORING ALL PATHS

Now (finally!) we can show how to satisfy Feynman's (and Nature's) complete command to the electron, *Explore ALL worldlines*. Up until now we have been just SAMPLING possible worldlines, for example pointing and clicking to choose a few worldlines between emission and detection. How can we sum the results of ALL paths that the electron can take between one point on an initial wavefunction and another point on the later wavefunction? One way to do this is to carry out the summation mathematically—a messy business. If you want to see professionals at work on this task, look into the first two Advanced Texts referenced in the *Suggested Readings in Quantum Mechanics* (Feynman and Hibbs page 42f and Schulman page 31f). A way to avoid summing all paths is to GUESS the result, try it out, and improve it until it works. That is what we will do here: a kind of “reverse engineering,” finding a technical way to achieve a prescribed end. We will use the results to describe a free particle (moving in a region of zero potential) as it explores ALL paths between an “emission point” on an initial wavefunction and a point on the later wavefunction.

What is our best first guess about how the initial wavefunction propagates forward in time? Well, the WAVEFUNCTION program did not do too bad a job (Figure 1). What did that program do? It rotated each arrow in the initial wavefunction along the *direct* worldline to a point on the final wavefunction, this rotation taking place at the rate $f = KE/h$. Between each initial dot and each final dot, this rotation used only the SINGLE direct worldline, rather than summing the results of all worldlines. But—hey!—it worked pretty well. So go with it. Our original trial process, used in the WAVEFUNCTION program, could be written as a word equation. Read the following equation from right to left; the multiplication symbol \times means “applied to” or “multiplied by.”).

$$\left(\begin{array}{l} \text{little arrow} \\ \text{at detection} \\ \text{event via} \\ \text{DIRECT path} \end{array} \right) = \left(\begin{array}{l} \text{rotate at rate} \\ f = KE/h \text{ along} \\ \text{DIRECT path} \end{array} \right) \times \left(\begin{array}{l} \text{one arrow} \\ \text{at emission} \\ \text{event} \end{array} \right) \quad \text{(trial version)} \quad (1)$$

But this is just our first crude try. When this job is done right—taking into account ALL paths—we expect the final little arrow to be not only rotated but also to be different in length from the initial arrow. We want to summarize the results of ALL paths with a single multiplier or procedure. A more general case might have the overall form (read right-to-left):

$$\left(\begin{array}{l} \text{little arrow} \\ \text{at detection} \\ \text{event via} \\ \text{ALL paths} \end{array} \right) = \left[\left(\begin{array}{l} \text{"finagle"} \\ \text{factor"} \end{array} \right) \times \left(\begin{array}{l} \text{rotate at rate} \\ f = KE/h \text{ along} \\ \text{DIRECT path} \end{array} \right) \right] \times \left(\begin{array}{l} \text{one arrow} \\ \text{at emission} \\ \text{event} \end{array} \right) \quad (2)$$

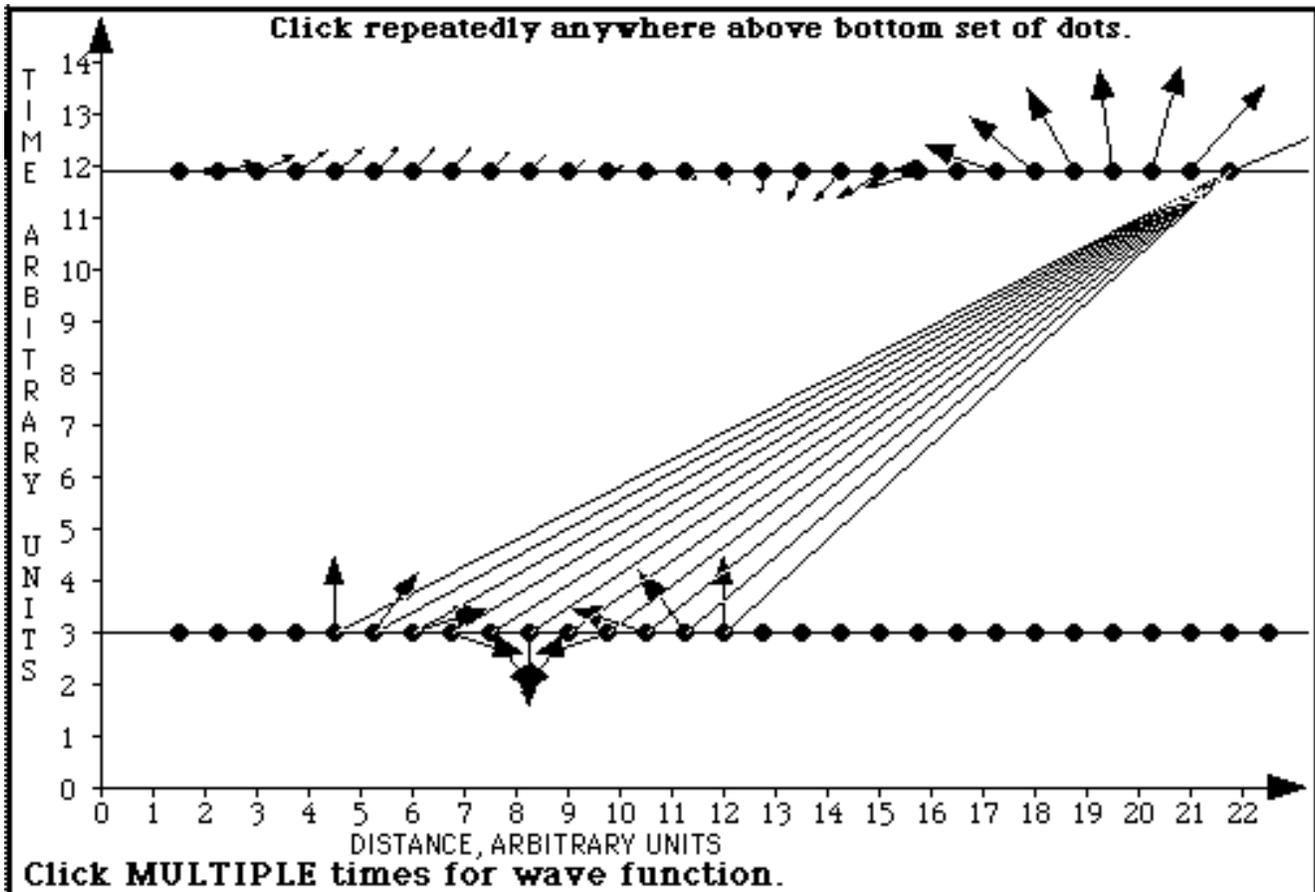


Figure 1. The WAVEFUNCTION program (that created this figure) uses the trial version of the propagator (Eq. 1 above) to propagate each initial arrow in the lower wavefunction along the SINGLE DIRECT worldline to a yield a final little arrow at a dot in the later wavefunction. The program then adds head-to-tail the little arrows from all the dots in the initial wavefunction to form the resulting arrow shown here at each dot in the later wavefunction. NEXT STEPS: Instead of using only the direct worldline, we will find a more accurate propagator (of the form given in Eqs. 2 and 3) that allows us to mimic the result of exploring ALL paths between each arrow on the initial wavefunction and a point on the final wavefunction. The final wavefunction is still assembled by summing all the propagator-calculated little arrows at each detection event head-to-tail to give Nature's (correct) resulting arrow at that point.

In equation (2) the “finagle factor” is something (as yet undetermined) that we stick in there to give the correct answer. We will see later that the “finagle factor” depends on the time lapse between emission and detection event. It also depends on the mass of the particle—and on some other stuff. Turns out that it does NOT depend on the spatial separation between emission and detection event—that is taken care of by the procedure in the bracket labeled “rotate at rate $f = E/h$ along DIRECT path.”

The square bracket on the right side of equation (2) constitutes the “thing” that changes the arrow at the emission event into the little arrow at the detection event. Equation (2) can be simplified by substituting a simpler-looking expression for the square bracket:

$$\begin{pmatrix} \text{little arrow} \\ \text{at detection} \\ \text{event D via} \\ \text{ALL paths} \end{pmatrix} = K(D,E) \times \begin{pmatrix} \text{one arrow} \\ \text{at emission} \\ \text{event E} \end{pmatrix} \quad (3)$$

(The words “*little arrow at detection event*” reminds us that the *resulting* arrow at the detection event comes from summing all the *little arrows* from every point in the initial wavefunction.) In equation (3) the expression $K(D,E)$ is a function that does two things to the initial arrow (duplicating what would happen if we summed the results of exploring ALL paths between emission E and detection D):

1. It rotates the initial arrow to the direction of the final arrow.
2. It changes the length of the initial arrow to the length of the little arrow at the detector.

The letters D and E are part of the function $K(D,E)$ to remind us that $K(D,E)$ depends on the differences in position and time between the Emission and Detection events. We want to find the function $K(D,E)$ that mimics for us the exploration along ALL PATHS which, taken together, change the initial arrow into the final arrow for a free particle.

II. THE PROPAGATOR

The function $K(D,E)$ is called the *propagator*. (Feynman and Hibbs call it the *Kernel*, which is why it has the symbol K .) The free-particle propagator tells us how an arrow from a single event in the initial wavefunction is changed into a little arrow at a single event in the final wavefunction when the particle explores ALL WORLDFINES between the two events.

The RESULTING arrow at the final point (the wavefunction at that point) is not just the result of the arrow at ONE point in the initial wavefunction. No, it is the result of ALL arrows in the initial wavefunction. To get the RESULTING arrow at the final point, we use the propagator to propagate EACH arrow in the initial wavefunction forward in time to yield a little arrow at that point in the final wavefunction. Then the computer ADDS UP all the little arrows at that final point, head-to-tail as usual, to reckon the resulting arrow at that point. Then the whole procedure is repeated for every other point in the final wavefunction. The WAVEFUNCTION program did this crudely and inaccurately, using the rotation along the single direct worldline and no change in length of the initial arrows in the process (Figure 1). In contrast, the propagator does the job right—telling us the result of exploring ALL paths between each initial and each final point in the absence of a potential.

III. THE UNIFORM WAVEFUNCTION

The FindProp software program helps you to try out various forms of the free-particle propagator until you find one that does the job. BUT—how will you know when the job is done correctly? We test our trial propagator by asking it to move forward in time an initial wavefunction that is the same everywhere in space. Our uniform initial wavefunction consists of a string of arrows of the same length all pointing upward (Figure 2). Uniformity is boring, because it does not change with time. And we will test our trial propagator by its ability to bore us—to construct a later wavefunction that also consists of a string of arrow of the same length as before, and all pointing upward.

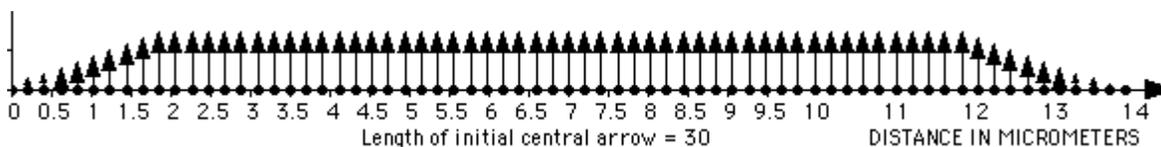


Figure 2. Initial wavefunction used to test our trial propagator functions. This wavefunction is uniform over a wide span of space. We demand that a successful propagator move this wavefunction forward in time without significant change. (The poor computer cannot deal with a row of arrows that goes on forever, so we taper the ends to create a gradual match between the uniform arrows and zero wavefunction. See below: Section XI Some Fine Points of Interest)

WHY should an initial wavefunction with equal arrows all pointing upward propagate forward in time in such a way that the arrows all remain the same? The equality of the squared magnitudes of these arrows implies an initial probability distribution everywhere the same in x . If the probability is everywhere the same in space, it has “nowhere to go”—no “empty” region into which it can “diffuse.” Because of the very wide extent of this initial wavefunction along the x direction, we expect that any “motion of probability” will leave local probability near the center unchanged for a long time. This analysis does not tell us that the arrows will also stay vertical with time, but we postulate this result as well. The program helps you to apply a trial propagator function between every dot in the initial wavefunction and each dot in the final wavefunction, allowing you to modify the propagator until the wavefunction does not change significantly with time.

TRY THIS ANALOGY: Suppose the entire universe is all at the same temperature (as it may be in the distant future!). Then, we expect, from that time forward the universe will *remain* at the same temperature. We cannot try this with a universe, so try it with your heated house, initially all at the same temperature. You open up the front and back doors on a windless, cold day. How long will it be before you notice a temperature change in the middle of the first floor? Guess: one minute. Then for ten seconds we expect the temperature to remain the same in the region near the middle of the first floor.

Temperature in this example is like probability in quantum mechanics. We would like to start with a wavefunction that is initially uniform over ALL space. But we cannot do that: the computer cannot do calculations with an infinite number of initial arrows! So instead think of a wavefunction that is initially uniform over a RANGE of space. “Initially uniform” means that at $t = 0$ all arrows are the same length and point in the same direction. Arrows of equal length mean (when this length is squared) equal initial probability of finding the electron in this region of space. How will this probability change? By leaking away at the edges, just as the heat in your house leaks out through the front and back doors. But in the center the probability will remain the same for a while.

So we expect the LENGTH of arrows in the center to remain the same for some time. Why should their DIRECTION remain the same also during this time? I am still working on the answer to this equation, but tell me what you think of the following one-word argument: Symmetry! For a uniform initial wavefunction infinite in spatial extent, why should the

initial upward-pointing arrows rotate clockwise, rather than counterclockwise, with time? You say: Because ALL the arrows rotate clockwise with time—that is what ordinary mechanical stopwatches *do*. Yes, but clockwise rotation is just a *convention*—what we *mean* by “clockwise.” We could just as well have chosen COUNTERclockwise rotation for the quantum stopwatch, and everything we have developed in this course would be the same. So the bottom line is: There is no reason why the arrows in a uniform initial wavefunction should rotate either clockwise or counterclockwise. So they do not rotate at all! Does this argument convince you? Me neither.

OBJECTION: Wait a minute! You are asking that the wavefunction arrows not change with time. Yet to construct each arrow of the later wavefunction, your propagator rotates the arrows along the direct worldline from each point on the initial wavefunction—then at each point in the later wavefunction adds head-to-tail all the arrows from the initial wavefunction. Make up your mind! Are the arrows rotating or not?

DISCUSSION: Think of your automobile moving along the street, behaving smoothly in public. Yet under the hood its crankshaft rotates rapidly, each piston oscillating up and down thousands of times a minute. Similarly, our unique wavefunction does not change with time; that is the smooth way it performs in public. But “underneath the hood” the quantum machinery spins along: The electron *Explores all worldlines*, its quantum clock rotating as it explores each worldline. The propagator summarizes for us this infinite number of explorations. But, lo and behold, when the propagator finishes its task and we summarize all the little arrows at each point in the final wavefunction, the wavefunction remains unchanged—for this particular uniform initial case. The form of the propagator that leads to THIS result is the one we seek.

After finding the correct propagator $K(D,E)$ for the free particle, we can apply it to ANY initial wavefunction (for a free particle) to tell us how this wavefunction changes with time. We will find that most wavefunctions DO change with time!

IV. FINDING THE PROPAGATOR

The software program **FindProp** (“Find Propagator”) starts with the simple propagator used in the WAVEFUNCTION program—one that simply rotates the arrow from each point in the initial wavefunction along the direct worldline to the point on the final wavefunction, without changing the length of this arrow. We call this the UNITY PROPAGATOR.

Figure 3 shows the result of applying the unity propagator. The initial uniform wavefunction displays itself along the bottom on the horizontal line $t = 0$. (Why does this initial wavefunction taper down to zero at either end? See below: Section XI Some Fine Points of Interest.) The upper three rows of arrows show the results of applying the unity propagator at three later times. The box across the upper right tells us that the arrows in these three later wavefunctions are much longer than the arrows in the initial wavefunction. Notice also that these later arrows all point at an inclined angle to the right instead of straight up. Finally, see that these arrows change length, getting longer for later times. We need to find a new propagator, with a “finagle factor” chosen so that all three of these errors are fixed -- allowing

the initial wavefunction to propagate forward unchanged. In what follows we use the FindProp software to find this correct propagator by trial and error.

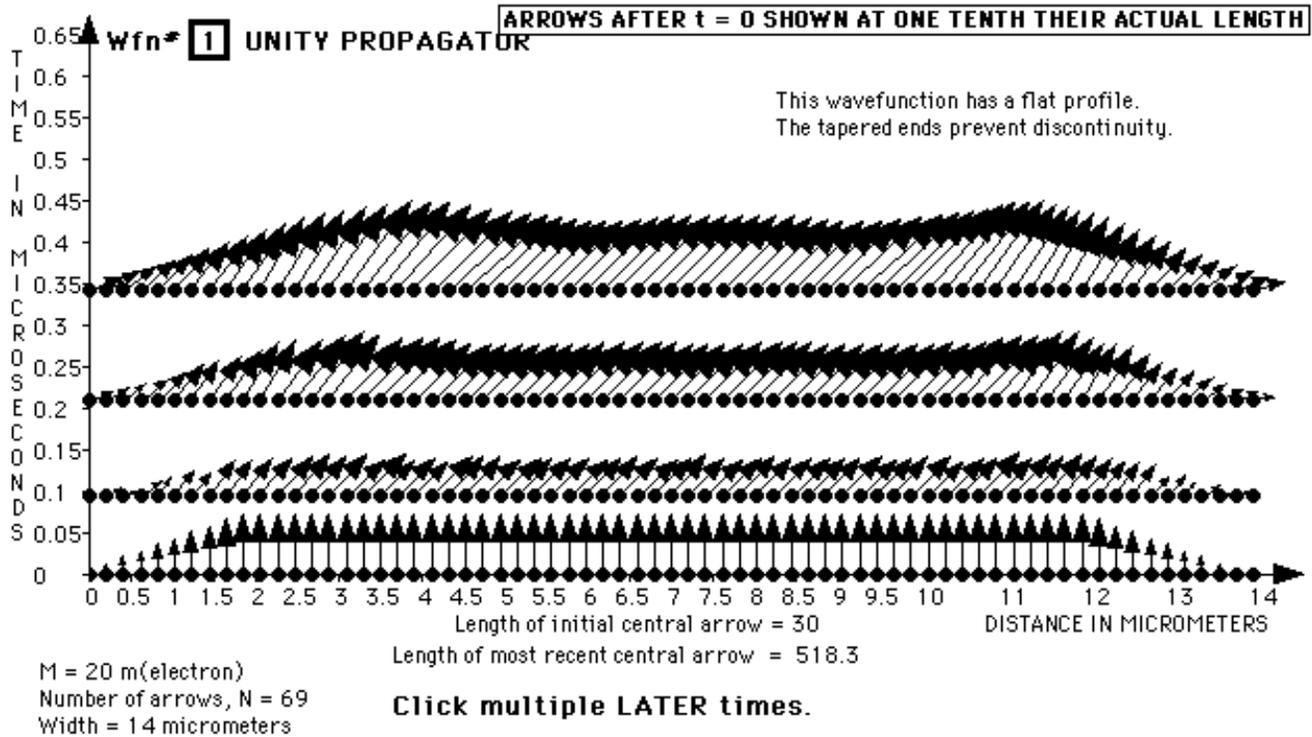


Figure 3. The result of applying the “unity propagator” (used in the WAVEFUNCTION program of Figure 1) for three later times to an initially uniform wavefunction. There are three things wrong with the resulting wavefunctions: (1) Arrows point in the wrong direction. (2) Arrows are too long (see box across the top). (3) Arrows grow bigger for later times. By trial and error we seek a revised propagator to make the later wavefunctions look like the initial wavefunction

DISCUSSION QUESTION: What is the source of the errors “wrong direction” and “too great length” of the arrows in the upper three rows in Figure 2? The answer is hidden in Figure 2 of the unit Worldlines for the Quantum Particle, reproduced here as Figure 4. The contribution of the direct path points in the direction of the boxed fat arrow at the center of the right-hand panel of Figure 4. The resultant arrow is rotated clockwise from the direction of this boxed fat arrow, as are the later-time arrows shown in Figure 3. Also, the resultant arrow is much longer than the arrow contributed by a single one of the arrows in the initial wavefunction.

After playing with the unity propagator, you are asked to switch to the TRIAL PROPAGATOR and change its features so that the arrows of the final wavefunction

1. point in the right direction (first task below), *and*
2. do not vary in length as time passes (second task) , *and*
3. have the same length as in the original uniform wavefunction (third task).

Any time you want, you can switch to the CORRECT PROPAGATOR and see how results should look (at least how well our poor program approximates how things should look!)

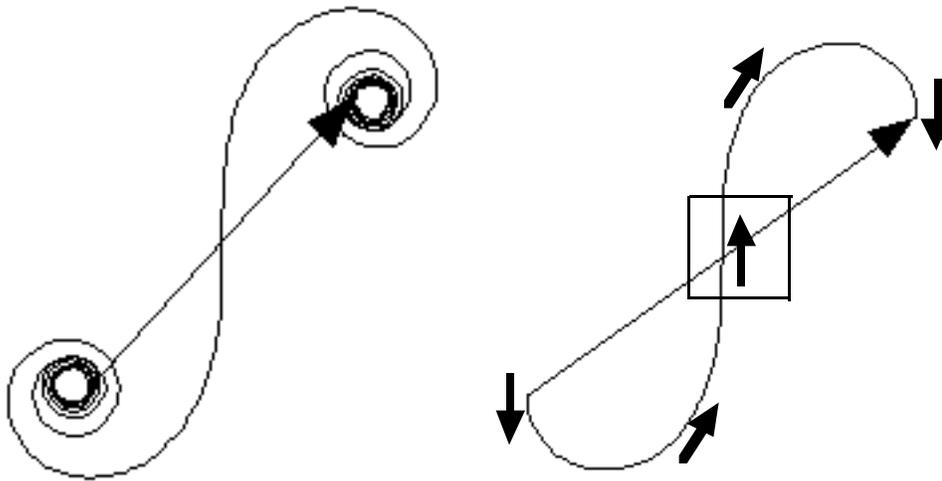


Figure 4. (Modified Figure 2 from the unit *Worldlines for Quantum Particles*) Resulting arrow for many alternative paths. The resulting arrow for a (nearly) complete Cornu spiral (left) from many worldlines is simplified (right) using contributions from only those worldlines along which the number of rotations differs by one-half-rotation or less from the direct worldline. The boxed small fat black arrow in the middle of the right-hand diagram is from the direct worldline. Notice relative size and direction of this and resulting arrow.

V. FIRST TASK: MAKE LATER ARROWS VERTICAL

Fire up the program and look at the initial wavefunction, which is uniform across most of the screen. (For now ignore the sloping tapered profile at each end.) Click above the initial wavefunction to see how well our UNITY propagator works, reproducing and extending the display shown in Figures 2 and 3.

To get a cleaner idea of what is going on, click on CHANGE DISPLAY and then on CENTER DOT. (Turn the PYRAMID on or off, as you wish.) Then click on OK. Now again click on later times to show only ONE resulting arrow of the later wavefunction, the arrow in the center.

Your first task is to make the arrows at the later wavefunction point in the right direction, namely straight up. You do this by telling the computer to apply an initial offset angle to the initial arrow before rotating it along the direct worldline.

Click on CHANGE PROPAGATOR, then on CHANGE TO TRIAL PROPAGATOR, then on CHANGE OFFSET ANGLE. Choose some angle other than zero, click OK, and try the new propagator. Continue changing the offset angle until arrows at later wavefunction all point upward. Answer the following question for the uniform initial wavefunction (red number 1 in upper left of screen):

Q1. For what value of the initial offset angle do the arrows in later wavefunction point in the same direction as the uniform arrows of initial wavefunction?

VI. SECOND TASK: MAKE LATER ARROWS INDEPENDENT OF TIME

We have found an initial angle offset that keeps all later arrows pointing in the correct direction (straight up). KEEP this value of the initial angle offset in what follows.

The second task is to make the arrows keep the same length, unchanging with time, for this particular uniform initial wavefunction. NOTE: The length does not have to be equal to 30, which is the length of the initial wavefunction arrows; that is the Third Task, below. For now we just want all LATER arrows to have approximately the same length.

To make later arrows not change with time, give the propagator a magnitude that itself changes with time so as to cancel the change in length of the final arrow that would otherwise occur.

How *should* the magnitude of the propagator change with time? We assume that this change is of the form

$$(\text{magnitude of propagator}) \propto (\text{time})^{\text{TIMEexponent}}$$

You will try out various values of the TIMEexponent to find the value that leads arrows in later wavefunctions to have a length that does not change with time. Click on the CHANGE PROPAGATOR button and check that under TRIAL PROPAGATOR the initial offset angle has the value you determined in Section V. Then click and on the CHANGE TIME FUNCTION button. Choose a TIMEexponent and try out the resulting propagator.

Q2. What value of the exponent yields a wavefunction whose central arrow has a constant length as time progresses?

VII. THIRD TASK: MAKE LATER ARROWS THE RIGHT LENGTH

So far we have changed the free-particle propagator so that the later arrows point in the right direction and have a length that does not change with time.

The third task is to make arrows in later wavefunctions not only unchanging in time but also equal in length to the arrows of the initial wavefunction. We do this by changing the coefficient that multiplies the propagator.

Click on CHANGE PROPAGATOR and check that the offset angle and time exponent have the values that you determined in Sections V and VI. Then click on CHANGE COEFFICIENT and choose a value that you feel will make arrows the same length in later wavefunction. Try this out until later arrows are the same length as the initial arrows, at least near the center of the wavefunction for the uniform initial wavefunction (red number 1 in upper left of screen).

Q3. For what coefficient of the propagator do arrows in the later wavefunctions have (approximately!) the same length as the arrows in the initial wavefunction?

VIII. FOURTH TASK: MAKE LATER ARROWS INDEPENDENT OF PARTICLE MASS

So far we have changed the free-particle propagator so that the later arrows point in the correct direction, do not change with time, and have the right length for the initial choice of particle mass.

Next, we want the propagator to give us the correct later wavefunction no matter what the mass of the particle. Of course, our primary client is the electron, but quantum mechanics must be correct for particles of different masses.

As before, we assume that the propagator changes with some power of the mass:

$$(\text{magnitude of propagator}) \propto (\text{mass})^{\text{MASSEXPOONENT}}$$

where the MASSEXPOONENT will probably be different from the exponent found for the time function.

Click on CHANGE PROPAGATOR and check that all your earlier results from Sections V, and VI and VII are entered in the TRIAL PROPAGATOR. Then click on CHANGE MASS FUNCTION and choose a MASSEXPOONENT for the mass function.

Change the particle mass starting with the CHANGE WAVEFUNCTION button. Use only values of the mass between 15 m(electron) to 100 m(electron). Try out various values of MASSEXPOONENT until you find the exponent that makes the later arrows (approximately!) the same length as the initial arrows, within this range of mass-values for the uniform initial wavefunction (red number 1 in upper left of screen).

Q4. For what value of the MASSEXPOONENT do arrows in later wavefunctions have the same length as those in the initial wavefunction, independent of particle mass?

IX. FIFTH TASK: APPLY THE PROPAGATOR TO OTHER WAVEFUNCTIONS

In deriving the free-particle propagator, we used an unrealistic wavefunction—one that is uniform over space. But now that we have derived the correct propagator, we can apply it to ANY initial free-particle wavefunction and watch this wavefunction develop in time. The fifth task is to do this and report the results.

Choose DEFAULT VALUES from the DEFAULT menu at the top of the initial screen. Then choose the CORRECT PROPAGATOR and apply it to wavefunctions 2 through 6. Change wavefunctions by clicking on the button NEXT WAVEFUNCTION. (For more on the sharp-ended uniform wavefunction #2, see the next section: FINE POINTS.)

Q5. For wavefunction #4, describe QUALITATIVELY how the wavefunction with the red number 4 propagates forward in time. In particular:

Q5A. Does the AVERAGE position move with time? (YES or NO)

AVERAGE position? For each time, is there a point from which the wavefunction “looks sort of the same” to the left of this point as looking to the right? That is the average position. Does the average position move with time?

Q5B. Does the PROFILE spread with time? (YES or NO)

PROFILE? As time passes, do the longish arrows spread farther away from the average position of question A?

Q5C. Do you notice any other effect worth reporting? (If yes, then please describe.)

Q6. Repeat questions Q5A, B, and C for wavefunction #5.

Q7. Repeat questions Q5A, B, and C for wavefunction #6.

X. SUMMARY OF RESULTS

Equations (2) and (3) in Section I give a kind of recipe for the propagator $K(D,E)$ between an emission event E and a detection event D :

$$K(D,E) = \left(\begin{array}{l} \text{"finagle factor"} \\ \text{between events} \\ E \text{ (emission) and} \\ D \text{ (detection)} \end{array} \right) \times \left(\begin{array}{l} \text{rotate at rate} \\ f = KE/h \text{ along} \\ \text{DIRECT path} \\ \text{from E to D} \end{array} \right) \quad (4)$$

The initial trial propagator, which we called the UNITY PROPAGATOR simply applied the procedure in the second parenthesis at the right of Equation (4), ignoring any instructions that might be contained in the first parenthesis. By trial and error, you now know a LOT more about the “finagle factor” in the first parenthesis of this equation. This finagle factor allows the single function $K(D,E)$ to mimic the result of exploring ALL paths between E and D .

Q8. Complete IN WORDS the following set of instructions, which are orders to the propagator $K(D,E)$ in propagating the initial arrow at the emission event E to the little final arrow at detection event D . Your instructions should include your results for all four tasks that you carried out to find the correct propagator. Be brief—the propagator is an impatient function!

Hey, Propagator! Here is what you gotta do. Between each point on the initial wavefunction and each point on the final wavefunction:

1. Before rotating the initial arrow along the direct worldline from E to D , give it an initial "offset angle" equal to . . .
2. In the process, multiply the length of the little arrow by the time-factor
3.
4.
-

If you want to see the mathematical equivalent to your word instructions, look at the Appendix below.

Q9. PREDICT what the **CORRECT** propagator will do to an initial wavefunction that is identical to the uniform wavefunction with tapered ends (Wavefunction #1) **EXCEPT** that every initial arrow is tilted 30 degrees from the vertical. **CHOOSE ONE** of the following:
Q9A: The initial wavefunction will propagate forward in time unchanged (in the same approximation that the original Wavefunction #1 propagates forward unchanged). If so, give an argument why this should be so. **OR**
Q9B: The initial wavefunction will change with time. If so, give an argument why this should be so.

XI SOME FINE POINTS OF INTEREST

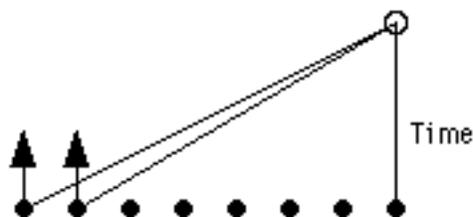
On the way through the four tasks of finding the free-particle propagator, you have probably noticed some strange things, a few of which we can explain here.

TAPERED WAVEFUNCTION. The “flat” initial wavefunction that we have used to set up the propagator has tapered ends. Without this taper, arrows in later wavefunctions show a fine-grained unevenness. Observe this by choosing **CORRECT PROPAGATOR**, then click once on the **NEXT WAVEFUNCTION** button. The resulting wavefunction (red number 2 in the box at the upper left of the screen) has sharp ends. When you click on later times, the later wavefunctions have arrows of uneven length, as you can verify. The sharp discontinuities at the ends spread rapidly through the wavefunction. This is a fundamental result of quantum mechanics: Sudden discontinuities in slope of the wavefunction profile lead to rapid changes of the wavefunction in time. (This is the basis of the Schroedinger equation, which relates *changes* of slope of the wavefunction to its *change* with time.)

INVALID WAVEFUNCTIONS. You may observe crazy wavefunctions at very short times after the initial wavefunction. This occurs especially when the particle mass is large. This effect is **NOT** fundamental physics. Rather, it comes from representing the wavefunction with discrete, evenly-spaced arrows.

The resulting arrow at the center of a later wavefunction (for example) is built up of contributions from all the arrows in the initial wavefunction. We rotate each of these arrows along the direct worldline between the initial arrow and the center point of the later wavefunction. Ordinarily these contributions add head-to-tail to form a Cornu Spiral, like Figure 4. However, a smooth Cornu Spiral results only when the difference in rotation is small between adjacent arrows in the initial wavefunction.

Now think of two worldlines stretching from two adjacent arrows near one end of the initial wavefunction to the central point in a later wavefunction, as shown in this figure.



If the time is very short, then the two slanting worldlines are nearly horizontal (covering a great distance in a short time), which describes a high velocity. High velocity means a large kinetic energy. (Greater mass also means a greater kinetic energy.) Large kinetic energy KE means a high rotation rate $f = KE/h$. So the arrows at the edge of the initial wavefunction make many rotations on their way to the central position of a later wavefunction.

Now if the DIFFERENCE between the rotations of adjacent initial arrows is great, the result is a jagged Cornu Spiral, or the destruction of the Cornu Spiral altogether. You can observe this directly, as follows:

In the CHANGE WAVEFUNCTION display, set the mass equal to 100 m(electron). Choose CENTER DOT in the CHANGE DISPLAY window. Then use the CORRECT PROPAGATOR and the flat initial wavefunction #1 (with tapered ends). You will see that, for short times, the Cornu Spiral is destroyed.

In summary, to avoid crazy wavefunctions due to the use of discrete arrows, we avoid setting the time interval to very small values, especially when the mass is large.

APPENDIX: The Free-Particle Propagator in Complex Notation

SUGGESTION: Review the introduction to complex numbers in the Appendix of an earlier unit in this manual, the one called The Electron in a Potential.

The arrow at the emission event can be represented as a complex number, a number which has magnitude and “direction,” the direction sometimes described by the word “phase.” The little arrow at the detection event that results from this emission arrow can also be represented as a complex number (magnitude and phase). And the propagator is a third complex number which tells the arrow at the emission event how to “shrink and turn” to yield the little arrow at the detection event.

As you have discovered, the propagator contains an initial offset angle (Task One above), is a function of time (Task Two), and also is a function of the mass of the particle (Task Four). Therefore the propagator is not a complex number but a **complex function** that changes with the time separation and particle mass. Equation (4) tells us that the propagator also has a factor that rotates the initial arrow along the direct worldline between emission and detection.

All of these conditions are satisfied by the following free-particle propagator:

$$K(D,E) = \left[\frac{m}{hT} \right]^{1/2} \exp\left(-\frac{i\pi}{4}\right) \exp\left(\frac{i\pi m X^2}{hT}\right) \quad (5)$$

Here X is the spatial distance between emission and detection events, and T is the time lapse between the two events. The function \exp means e (the base of natural logarithms) with an exponent given by the parenthesis that follows it. So for those with sharp eyesight, equation (5) can be rewritten in the form:

$$K(D,E) = \left[\frac{m}{hT} \right]^{1/2} e^{-\frac{i\pi}{4}} e^{\frac{i\pi m X^2}{hT}} \quad (6)$$

Now, have a look at equations (5) and (6). Those with some acquaintance with complex numbers will notice the following:

1. The imaginary number i in the exponents tell us that these factors represent rotations rather than an “exponential” increase or decrease of magnitude.
2. The exponent $-i\pi/4$ represents an offset angle of MINUS one-eighth turn, or minus 45 degrees that you derived (did you?) in Task One.
3. The inverse square root $T^{-1/2}$ in the initial square bracket is the result you derived in Task Two.
4. The square root $m^{+1/2}$ in the initial square bracket is the result you derived in Task Four.
5. The argument of the second exponential function can be written as the imaginary number i times the quantity:

$$\frac{\pi m X^2}{hT} = \frac{\pi m X^2}{hT^2} T = \frac{2\pi m X^2}{2hT^2} T = 2\pi \left(\frac{KE}{h} \right) T = 2\pi f T = 2\pi \left(\begin{array}{c} \text{number of} \\ \text{rotations} \\ \text{along direct} \\ \text{worldline} \end{array} \right) \quad (7)$$

Here KE is the *classical* kinetic energy of the particle along the direct worldline between emission and detection. And $KE/h = f$ is the frequency f of rotation of the quantum clock for a free electron. (See equation 3 of the Free Electron unit). Since T is the time between emission and detection, therefore fT is the number of rotations along this direct worldline. Why is there a 2π in the last expression on the right of equation (7)? Because the imaginary exponent of e measures angles in *radians* rather than number of rotations. And there are 2π radians per rotation.

NOTE: You can also show that the exponent can be written as i times the quantity:

$$\frac{\pi m X^2}{hT} = \frac{2\pi S}{h} = \frac{S}{(h/2\pi)} = \frac{S}{\hbar}$$

where S is the *classical action* computed along the direct worldline and $\hbar \equiv h/(2\pi)$, called “h-bar,” has wide use in physics. To demonstrate this form, see equation (17) in the unit on The Principle of Least Action for the case of zero potential ($g = 0$).

So the complex representation of the free-particle propagator in equations (5) and (6) fulfills all the conditions we derived for this propagator by our trial-and-error method using the FindProp software.

Copyright © 2000 Edwin F. Taylor