Project A

Global Positioning System

There is no better illustration of the unpredictable payback of fundamental science than the story of Albert Einstein and the Global Positioning System [GPS] . . . the next time your plane approaches an airport in bad weather, and you just happen to be wondering "what good is basic science," think about Einstein and the GPS tracker in the cockpit, guiding you to a safe landing.

—Clifford Will

1 Operation of the GPS

Do you think that general relativity concerns only events far from common experience? Think again! Your life may be saved by a hand-held receiver that "listens" to overhead satellites, the system telling you where you are at any place on Earth. In this project you will show that this system would be useless without corrections provided by general relativity.

The Global Positioning System (GPS) includes 24 satellites, in circular orbits around Earth with orbital period of 12 hours, distributed in six orbital planes equally spaced in angle. Each satellite carries an operating atomic clock (along with several backup clocks) and emits timed signals that include a code telling its location. By analyzing signals from at least four of these satellites, a receiver on the surface of Earth with a built-in microprocessor can display the location of the receiver (latitude, longitude, and altitude). Consumer receivers are the approximate size of a hand-held calculator, cost a few hundred dollars, and provide a position accurate to 100 meters or so. Military versions decode the signal to provide position readings that are more accurate—the exact accuracy a military secret. GPS satellites are gradually revolutionizing driving, flying, hiking, exploring, rescuing, and map making.

Airports use one GPS receiver at the control tower and one on the approaching airplane. The two receivers are close together, which cancels errors due to propagation of signals between each receiver and overhead satellites. It also cancels the "jitter" intentionally introduced into the satellite signal to make civilian receivers less accurate than military receivers.

Operation of the Global Positioning System

The goal of the Global Positioning System (GPS) is to determine your position on Earth in three dimensions: east-west, north-south, and vertical (longitude, latitude, and altitude). Signals from three overhead satellites provide this information. Each satellite sends a signal that codes where the satellite is and the time of emission of the signal. The receiver clock times the reception of each signal, then subtracts the emission time to determine the time lapse and hence how far the signal has traveled (at the speed of light). This is the distance the satellite was from you when it emitted the signal. In effect, three spheres are constructed from these distances, one sphere centered on each satellite. You are located at the single point at which the three spheres intersect.

Of course there is a wrinkle: The clock in your hand-held receiver is not nearly so accurate as the atomic clocks carried in the satellites. For this reason, the signal from a fourth overhead satellite is employed to check the accuracy of the clock in your hand-held receiver. This fourth signal enables the hand-held receiver to process GPS signals as though it contained an atomic clock.

Signals exchanged by atomic clocks at different altitudes are subject to general relativistic effects described using the Schwarzschild metric. Neglecting these effects would make the GPS useless. This project analyzes these effects.

As a result, measurement of the *relative* position of control tower and airplane is accurate to 1 or 2 meters. This configuration of receivers permits blind landing in any weather. Runway collisions can also be avoided by using this system to monitor positions of aircraft on the ground (a task impossible for the electromagnetic signals of radar).

The timing accuracy required by the GPS is so great that general relativistic effects are central to its performance. First, clocks run at different rates when they are at different distances from a center of gravitational attraction. Second, both satellite motion and Earth rotation must be taken into account; neither the moving satellite nor Earth's surface corresponds to the stationary spherical shell described in Chapter 2. In this project you will investigate these effects.

Your challenge in this project (and in all later projects) is to respond to the numbered queries. (Query 1 for this project appears on page A-4.) Typically, a query contains several related questions. Answer the queries in order, or as assigned to you, or skip to those that interest you the most.

2 Stationary Clocks

Earth rotates and is not perfectly spherical, so, strictly speaking, the Schwarzschild metric does not describe spacetime above Earth's surface. But Earth rotates slowly and the Schwarzschild metric is a good approximation for purposes of analyzing the Global Positioning System.

$$d\tau^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \frac{dr^2}{1 - \frac{2M}{r}} - r^2 d\phi^2 \qquad [1]$$

Apply this equation twice, first to the orbiting satellite clock and second to a clock fixed at Earth's equator and rotating as Earth turns. Both the Earth clock and the satellite clock travel at constant radius around Earth's center.

So $dr = 0$ for each clock. Divide the Schwarzschild metric through by the square of the far-away time dt^2 to obtain, for either clock,

$$\left(\frac{d\tau}{dt}\right)^2 = \left(1 - \frac{2M}{r}\right) - r^2\left(\frac{d\phi}{dt}\right)^2 = \left(1 - \frac{2M}{r}\right) - v^2 \qquad [2]$$

Here $d\tau$ is the wristwatch time between ticks of either clock and $v = r\,d\phi/dt$ is the tangential velocity along the circular path of the same clock as measured by the bookkeeper using far-away time measurement. Write down equation [2] *first* for the satellite, using $r = r_{\text{satellite}}$, $v = v_{\text{satellite}}$, and $d\tau = dt_{\text{satellite}}$ between ticks of the satellite clock, *second* for the Earth clock, using $r = r_{\text{Earth}}$, $v = v_{\text{Earth}}$ and time $d\tau = dt_{\text{Earth}}$ between ticks of the Earth clock, all these for the same time lapse dt on the far-away clock. Divide corresponding sides of these two equations to obtain the squared ratio of time lapses recorded on the satellite and earth clocks:

$$\left(\frac{dt_{\text{satellite}}}{dt_{\text{Earth}}}\right)^2 = \frac{1 - \dfrac{2M}{r_{\text{satellite}}} - v_{\text{satellite}}^2}{1 - \dfrac{2M}{r_{\text{Earth}}} - v_{\text{Earth}}^2} \qquad [3]$$

The general relativistic effects we study are small. How small? Small compared to what? When *must* one use exact general relativistic expressions? When are approximations good enough? These questions are so central to the analysis that it is useful to begin with a rough estimate of the size of the expected effects, not worrying for now about the crudeness of this approximation.

Start by ignoring the motions of satellite clock and Earth surface clock. Ask instead what the difference in clock rates will be for *stationary* clocks at these two radii. Then equation [3] can be written

$$\frac{dt_{\text{satellite}}}{dt_{\text{Earth}}} \approx \frac{\left(1 - \dfrac{2M}{r_{\text{satellite}}}\right)^{1/2}}{\left(1 - \dfrac{2M}{r_{\text{Earth}}}\right)^{1/2}} = \left(1 - \frac{2M}{r_{\text{satellite}}}\right)^{1/2}\left(1 - \frac{2M}{r_{\text{Earth}}}\right)^{-1/2} \qquad [4]$$

You will show in Query 7 that the radius of a 12-hour circular orbit is about 26.6×10^6 meters from Earth's center. You will find values for the radius and mass of Earth among the constants inside the back cover.

We now make first use of an approximation that appears repeatedly in this project:

$$(1 + d)^n \approx 1 + nd \qquad \text{provided} \qquad |d| \ll 1 \qquad \text{and} \qquad |nd| \ll 1 \qquad [5]$$

Here the two vertical lines mean "absolute value."

Approximation [5] is accurate for any real (positive or negative, integer or fractional) value of the exponent *n* provided the absolute values of *d* and *nd* are both very much less than unity. Equation [5] is used so often in this book that it is rewritten for general reference as equation [E] on the last page of the book.

QUERY 1 **Formula: Clock rate difference due to height.** Apply approximation [5] to the two parenthetical expressions on the right of equation [4]. Multiply out the result to show that

$$\frac{dt_{\text{satellite}}}{dt_{\text{Earth}}} \approx 1 - \frac{M}{r_{\text{satellite}}} + \frac{M}{r_{\text{Earth}}} - \frac{M}{r_{\text{satellite}}}\frac{M}{r_{\text{Earth}}} \qquad [6.\ \text{for}\ v = 0]$$

QUERY 2 **Improved approximation.** What are the approximate values of M/r_{Earth} and $M/r_{\text{Satellite}}$? Make an argument that the last term on the right of [6] can be neglected in comparison with the other terms on the right, leading to the result for stationary satellite and Earth clocks:

$$\frac{dt_{\text{satellite}}}{dt_{\text{Earth}}} \approx 1 - \frac{M}{r_{\text{satellite}}} + \frac{M}{r_{\text{Earth}}} \qquad [7.\ \text{for}\ v = 0]$$

QUERY 3 **Numerical approximation.** Substitute numbers into equation [7] and find the numerical value of *b* in the following equation:

$$\frac{dt_{\text{satellite}}}{dt_{\text{Earth}}} \approx 1 + b \qquad [8.\ \text{for}\ v = 0]$$

The number represented by *b* in equation [8] is an estimate of the fractional difference in rates between stationary clocks at the position of the satellite and at Earth's surface. Is this difference negligible or important to the operation of the GPS? Suppose the timing of a satellite clock is off by 1 nanosecond (10^{-9} second). In 1 nanosecond a light signal (or a radio wave) propagates approximately 30 centimeters, or about one foot. So a difference of, say, hundreds of nanoseconds will create difficulties.

QUERY 4 **Synchronization discrepancy after one day.** There are 86,400 seconds in one day. To one significant figure, the satellite clocks and Earth clock go out of synchronism by about 50 000 nanoseconds per day due to their difference in altitude alone. Find the correct value to three-digit accuracy.

The satellite clock will "run fast" by something like 50 000 nanoseconds per day compared with the clock on Earth's surface due to position effects alone. Clearly general relativity is needed for correct operation of the Global Positioning Satellite System! On the other hand, the *fractional* difference between clock rates at the two locations (at least the fraction due to difference in radius) is small.

In addition to effects of position, we must include effects due to motion of satellite and Earth observer. In which direction will these effects influence the result? The satellite clock moves faster than the clock revolving with Earth's surface. But special relativity tells us that (in an imprecise summary) "moving clocks run slow." This prediction agrees with the negative sign of v^2 in equations [2] and [3]. Therefore we expect the effect of motion to *reduce* the amount by which the satellite clock runs fast compared to the Earth clock. In brief, when velocity effects are taken into account, we expect the satellite clock to run faster than the Earth clock by *less* than the estimated 50 000 nanoseconds per day. We will need to check our final result against this prediction.

3 Clock Velocities

Now we need to take into account the velocities of Earth and satellite clocks to apply the more complete equation [3] to our GPS analysis. What are the values of the clock velocities v_{Earth} and $v_{\text{satellite}}$ in this equation, and who measures these velocities? For the present we find the simplest measure of these velocities, using speeds calculated from Euclidean geometry and Newtonian mechanics. Newton uses a fictional "universal" time t, so Newtonian results will have to be checked later in a more careful analysis.

QUERY 5 **Speed of a clock on the equator.** Earth's center is in free float as Earth orbits Sun and rotates on its axis once per day (once per 86 400 seconds). With respect to Earth's center, what is the speed v_{Earth} of a clock at rest on Earth's surface at the equator? Use Newtonian "universal" time t. Express your answer as a fraction of the speed of light.

What is the value of the speed $v_{\text{satellite}}$ of the satellite? Newton tells us that the acceleration of a satellite in a circular orbit is directed toward the center and has the magnitude v_{conv}^2/r, where v_{conv} is measured in conventional units, such as meters per second. The satellite mass m multiplied by this acceleration must be equal to Newton's gravitational force exerted by Earth:

$$\frac{GM_{\text{kg}}m}{r_{\text{satellite}}^2} = \frac{mv_{\text{conv}}^2}{r_{\text{satellite}}} \qquad [9]$$

Equation [9] provides one relation between the velocity of the satellite and the radius of its circular orbit. A second relation connects satellite velocity and orbit radius to the period of one revolution. This period T is 12 hours for GPS satellites:

$$v_{\text{conv}} = \frac{2\pi r_{\text{satellite}}}{T_{\text{seconds}}} \qquad [10]$$

QUERY 6 **Geometric units.** In equations [9] and [10] convert the mass M to units of meters and convert satellite speed to a fraction of the speed of light. Leave T in units of seconds. Then eliminate the radius $r_{\text{satellite}}$ between these two equations to find an expression for $v_{\text{satellite}}$ in terms of M and T_{seconds} and numerical constants.

4 The Final Reckoning

QUERY 7 **Satellite radius and speed.** Find the numerical value of the speed $v_{\text{satellite}}$ (as a fraction of the speed of light) for a satellite in a 12-hour circular orbit. Find the numerical value of the radius $r_{\text{satellite}}$ for this orbit—according to Newton and Euclid.

Now we have numerical values for all the terms in equation [3] and can approximate the difference in rates for satellite clocks and Earth clocks.

QUERY 8 **Formula: Clock rate difference.** Take the square root of both sides of equation [3]. Do not substitute numerical values yet. Rather, for both numerator and denominator in the resulting equation, use the approximation [5], as follows, In the numerator, set

$$d = -\frac{2M}{r_{\text{satellite}}} - v_{\text{satellite}}^2 \qquad [11]$$

In the denominator, use the same expression for d but with “Earth” as the subscripts. Carry out an analysis similar to that in Query 2 to preserve only the important terms. Show that the result is

$$\frac{dt_{\text{satellite}}}{dt_{\text{Earth}}} \approx 1 - \frac{M}{r_{\text{satellite}}} - \frac{v_{\text{satellite}}^2}{2} + \frac{M}{r_{\text{Earth}}} + \frac{v_{\text{Earth}}^2}{2} \qquad [12]$$

QUERY 9 **Numerical clock rate difference.** Substitute values for the various quantities in equation [12]. Result: To *two* significant figures, the satellite clock appears to run faster than the Earth clock by approximately 39 000 nanoseconds per day. Give your answer to *three* significant figures.

Section 2 described the difference in clock rates due only to difference in altitude. We predicted at the end of Section 2 that the full general relativistic treatment would lead to a *smaller* difference in clock rates than the altitude effect alone. Your result for Query 9 verifies this prediction. In the following section we examine some of the other approximations made in the analysis.

A practical aside: When Carroll O. Alley was consulting with those who originally designed the Global Positioning System, he had a hard time convincing them not to apply *twice* the correction given in equation [12]: first to account for the difference in clock rates at the different altitudes and second to allow for the blue shift in frequency for the signal sent downward from satellite to Earth. There is only one correction; moreover there is no way to distinguish what is the “cause” of this correction. Hear what Clifford Will has to say on the subject, as he describes the difference in rates between one clock on a tower and a second clock on the ground:

> *A question that is often asked is, Do the intrinsic rates of the emitter and receiver or of the clock change, or is it the light signal that changes frequency during its flight? The answer is that it doesn’t matter. Both descriptions are physically equivalent. Put differently, there is no operational way to distinguish between the two descriptions. Suppose that we tried to check whether the emitter and the receiver agreed in their rates by bringing the emitter down from the tower and setting it beside the receiver. We would find that indeed they agree. Similarly, if we were to transport the receiver to the top of the tower and set it beside the emitter, we would find that they also agree. But to get a gravitational red shift, we must separate the clocks in height; therefore, we must connect them by a signal that traverses the distance between them. But this makes it impossible to determine unambiguously whether the shift is due to the clocks or to the signal. The observable phenomenon is unambiguous: the received signal is blue shifted. To ask for more is to ask questions without observational meaning. This is a key aspect of relativity, indeed of much of modern physics: we focus only on observable, operationally defined quantities, and avoid unanswerable questions.*

5 Justifying the Approximations

We calculated the speed of a satellite in circular orbit and the speed of the clock on Earth’s surface using Euclidean geometry and Newtonian mechanics with its “universal time.” Now, the numerator in each expression for speed, namely $rd\phi$, is the same for Euclidean geometry as for Schwarzschild geometry because of the way we defined r in Schwarzschild spacetime. However, the time dt in the denominator of the speed is not the same for Newton as for Schwarzschild. In particular, the derivation of equation [3] assumes that the speeds in that equation are to be calculated using changes in far-away time dt. Think of a spherical shell constructed at the radius of the satellite orbit and another “shell” that is the surface of Earth. Then our task boils down to estimating the difference between far-away time dt and shell time dt_{shell} in each case, which can be done using our equation [C] in Selected Formulas at the end of this book.

$$dt_{\text{shell}} = \left(1 - \frac{2M}{r}\right)^{1/2} dt \qquad [13]$$

QUERY 10 **Effect of using Schwarzschild far-away time instead of Newton's "universal time."** Use equation [13] and the approximation equation [5] to set up an approximate relation between the two measures of velocity:

$$\frac{rd\phi}{dt} \approx \frac{rd\phi}{dt_{\text{shell}}}(1-q) \qquad [14]$$

where q is a small number. Find an algebraic expression for q. Then find numerical values of q both for Earth's surface and at the orbit radius of the satellite. Use these results to estimate the difference that changed velocity values will make in the numerical result of Query 9. Is this difference significant?

Two Notes

Note 1: The approximate analysis in this project also assumed that the radius $r_{\text{satellite}}$ of the circular orbit of the satellite is correctly computed using Newtonian mechanics. The Schwarzschild analysis of circular orbits is carried out in Chapter 4. When you have completed that chapter, you will be able to show that this approximate analysis is sufficiently accurate for our purposes.

Note 2: Our analysis assumed the speed v_{Earth} of the Earth clock to be that of the speed of the equator. One might expect that this speed-dependent correction would take on different values at different latitudes north or south of the equator, going to zero at the poles where there is no motion of the Earth clock due to rotation of Earth. In practice there is no latitude effect because Earth is not spherical; it bulges a bit at the equator due to its rotation. The smaller radius at the poles increases the M/r_{Earth} term in equation [12] by the same amount that the velocity term decreases. The outcome is that our calculation for the equator applies to all latitudes.

6 Summary

A junior traveler, making her first trip by train from the United States into Mexico, sees the town of Zacatecas outside her window and reassures herself by the marginal note in the guidebook that this ancient silver-mining town is 1848 kilometers from San Diego, California, and 1506 kilometers from New Orleans, Louisiana. On a surface, two distances thus suffice to fix location. But in space it is three. Find those three distances, to each of three nearest satellites of the Global Positioning System, by finding the time taken by light or radio pulse to come from each satellite to us. Simple enough! Or simple as soon as we correct, as we must and as we have, for the clock rates at each end of the signal path. (1) General relativity predicts that both the relative altitudes and the relative speeds of satellite and Earth clocks affect their relative rates. (2) The clock in the hand-held receiver on Earth is far less accurate than the atomic clock in each satellite, so the signal from a fourth satellite is employed to correct the Earth clock. With these corrections, we can use the Global Positioning System to locate ourselves anywhere on Earth with an uncertainty of only a few meters.

7 References and Acknowledgments

Initial quote from “A Tale of Einstein, Congress and Safe Landings” by Clifford Will, an online posting by the Editorial Services, Washington University, St. Louis.

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Quote about clocks on the ground and on a tower: Clifford M. Will, *Was Einstein Right? Putting General Relativity to the Test*, Second Edition, Basic Books / Perseus Group, New York, 1993, pages 49–50.