Chapter 9. Orbiting the Black Hole

- *As I approach a black hole from far away, how can I put my spaceship into a circular orbit?*

- *How can I transfer from one circular orbit to another one?*

- *Why am I uncomfortable in some orbits near a black hole?*

- *Can I enter a circular orbit without firing a rocket?*

- *How do I move a probe from a circular orbit inward across the event horizon?*

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CHAPTER

9

Orbiting the Black Hole

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*I want to know how God created this world. I am not interested
in this or that phenomenon, in the spectrum of this or that
element. I want to know his thoughts. The rest are details.*

****

*What really interests me is whether God could have created
the world any differently; in other words, whether the
requirement of logical simplicity admits a margin of freedom.*

—Albert Einstein

9.1 ■ OBSERVE THE BLACK HOLE FROM A SEQUENCE OF CIRCULAR ORBITS

The sequence of orbits in our exploration plan

Chapter 8 introduced circular orbits of a free stone around a black hole. The
present chapter describes how the captain of an approaching spaceship can
insert it into a circular orbit, then transfer to progressively smaller circular
orbits in order to get closer looks at the black hole. Our exploration program
includes several maneuvers:

Observe the
black hole from
circular orbits.

EXPLORATION PROGRAM FOR THE BLACK HOLE

Step 1. Insert the approaching spaceship into a stable circular orbit at
$r = 20M$.

Exploration program

Step 2. Transfer a probe from this initial orbit to the innermost stable circular
orbit at $r_{\text{ISCO}} = 6M$.

Step 3. Transfer the probe from the ISCO to an *unstable* circular orbit at
$r = 4M$.

Step 4. Tip the probe off the unstable circular orbit at $r = 4M$ so that it
spirals inward across the event horizon.

46   To describe this sequence of orbits, use equations from previous chapters,
47   summarized here in global rain coordinates, T, r, ϕ. Both the unpowered
48   spaceship and the unpowered probe move in the same way as a free stone.

Free motion

49   **GENERAL FREE MOTION OF UNPOWERED SPACESHIP OR PROBE**

$$\frac{E}{m} \equiv \left(1 - \frac{2M}{r}\right) \frac{dT}{d\tau} - \left(\frac{2M}{r}\right)^{1/2} \frac{dr}{d\tau} \qquad \text{(free: (35) in Sec. 7.5)} \quad (1)$$

$$\frac{L}{m} \equiv r^2 \frac{d\phi}{d\tau} \qquad \text{(free: (10) in Sec. 8.2)} \quad (2)$$

$$\frac{E_{\text{shell}}}{m} = \frac{1}{\left(1 - v_{\text{shell}}^2\right)^{1/2}} = \frac{E/m}{\left(1 - \dfrac{2M}{r}\right)^{1/2}} \qquad \text{(free: (12) in Sec. 6.3)} \quad (3)$$

$$\left(\frac{dr}{d\tau}\right)^2 = \left(\frac{E}{m}\right)^2 - \left(\frac{V_{\text{L}}(r)}{m}\right)^2 \qquad \text{(free: (19) in Sec. 8.4)} \quad (4)$$

$$\left(\frac{V_{\text{L}}(r)}{m}\right)^2 \equiv \left(1 - \frac{2M}{r}\right)\left(1 + \frac{L^2}{m^2 r^2}\right) \qquad \text{(free: (18) in Sec. 8.4)} \quad (5)$$

Motion in a
circular orbit

50   **CIRCULAR-ORBIT MOTION OF UNPOWERED SPACESHIP OR PROBE ($r > 3M$)**

$$\left(\frac{L}{m}\right)^2 = \frac{Mr^2}{r - 3M} \qquad \text{(circle: (25) in Sec. 8.5)} \quad (6)$$

$$r = \frac{L^2}{2m^2 M}\left[1 \pm \left(1 - \frac{12 m^2 M^2}{L^2}\right)^{1/2}\right] \qquad \text{(circle: (26) in Sec. 8.5)} \quad (7)$$

$$\frac{E}{m} = \frac{r - 2M}{[r(r - 3M)]^{1/2}} \qquad \text{(circle: (32) in Sec. 8.5)} \quad (8)$$

$$\frac{E_{\text{shell}}}{m} = \left(\frac{r - 2M}{r - 3M}\right)^{1/2} \qquad \text{(circle: (33) in Sec. 8.5)} \quad (9)$$

$$v_{\text{shell}}^2 = \frac{M}{r - 2M} \qquad \text{(circle: (31) in Sec. 8.5)} \quad (10)$$

51   Figure 1 previews some kinds of orbits we discuss in this chapter.

9.2 ■ INSERT THE APPROACHING SPACESHIP INTO A CIRCULAR ORBIT

53   *Approach from far away and enter a circular orbit.*

Insert into a
circular orbit.

54   How does the captain insert her approaching spaceship into an initial circular
55   orbit from which to observe the black hole? Here's one possible method: While
56   still far from the black hole, the captain uses speed- and direction-changing
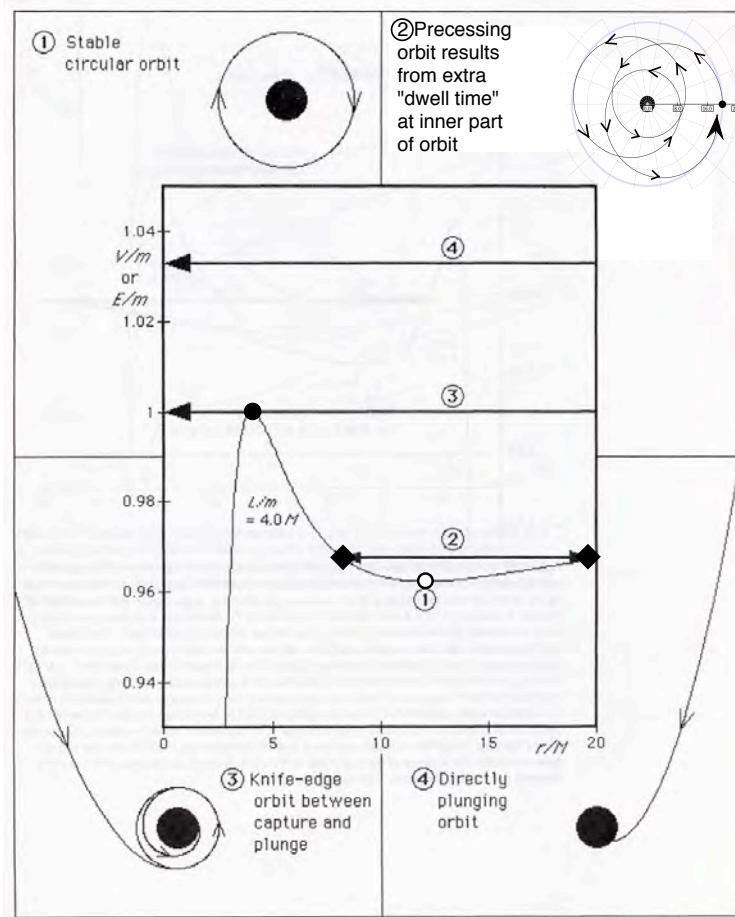
FIGURE 1   Preview: Some kinds of orbits discussed in this chapter, shown here for a single value of map angular momentum L/m but several different values of map energy E/m. A glance at the central plot allows us to make quick predictions about the motion of a stone that orbits or is captured by a black hole. Four different energies numbered on this plot correspond to orbits that appear in the four outer corners of the figure. Adapted from Misner, Thorne, and Wheeler.

rocket thrusts to put the spaceship into a free-fall insertion orbit whose minimum r-value matches that of the desired circular orbit (Figure 2). At that minimum, when the spaceship moves tangentially for an instant, the captain fires a rocket to slow down the spaceship to the tangential speed of the stable circular orbit at that r.

Insertion orbit

With what values of map E/m and L/m will an unpowered spaceship approaching from far away end up moving tangentially for an instant at the desired r-coordinate? To find out, substitute (5) into (4), set $dr/d\tau = 0$, and solve the resulting equation for L/m:

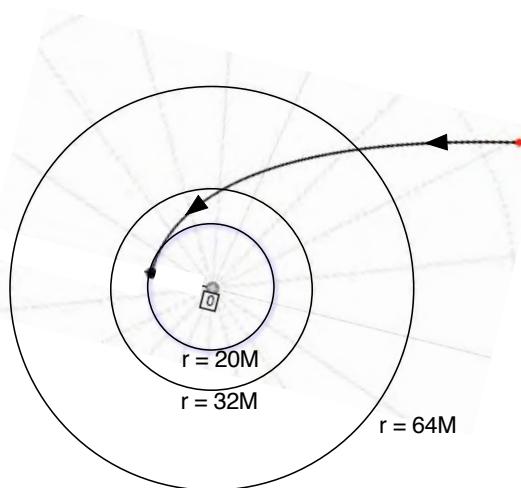**FIGURE 2**   Insertion orbit for unpowered spaceship that approaches from far away. At the instant of tangential motion at $r = 20M$, the spaceship fires a tangential rocket thrust to reduce the locally-measured shell velocity to that for a circular orbit (Figure 3).

$$\frac{L}{m} = \pm r \left[\frac{(E/m)^2}{1 - (2M/r)} - 1 \right]^{1/2} \qquad \text{(tangential motion)} \qquad (11)$$

66

67 The $\pm$ sign in (11) distinguishes between two possible directions of motion at
68 the r-value in equation (11). We choose positive angular momentum—that is,
69 in the counterclockwise direction of increasing ϕ. Equation (11) is valid when
70 $dr/d\tau = 0$, including turning points of all orbits as well as everywhere along a
71 circular orbit.

Choose circular
orbit at $r = 20M$.

72   The captain chooses her circular orbit at $r = 20M$. While still far from the
73 black hole, she maneuvers the incoming spaceship to move with
74 arbitrarily-chosen map energy $E/m = 1.001$ and the positive value of L/m that
75 results from equation (11)—both entered in Table 1. Then she turns off the
76 rockets and lets the spaceship coast. Figure 2 shows the resulting orbit, which
77 corresponds to the incoming horizontal arrow at $E/m = 1.001$ in Figure 3.

78   **DEFINITION 1. Subscripts in Table 1**

Definitions:
Subscripts
in Table 1

79   Here are definitions of the subscripts in the left-hand column of Table 1.
80   All definitions describe the motion of a free stone or unpowered
81   spaceship or unpowered probe.

82   **insert:** for free motion from far away to instantaneous tangential motion at r

83   **circle:** for free motion in a circular orbit at r

84 **transfer:** for free motion that is instantaneously tangential at both values of r

85   **shell:** for values measured in the local inertial frame at r

TABLE 9.1   Numerical values at $r = 20M$ and $r_{\mathrm{ISCO}} = 6M$

Values of	$r = 20M$	$r_{\mathrm{ISCO}} = 6M$
$(L/m)_{\mathrm{insert}}$	6.733 036 31M	———
$(E/m)_{\mathrm{insert}}$	1.001	———
$v_{\mathrm{x,shell,insert}}$	0.319 056 897	———
$(L/m)_{\mathrm{circle}}$	4.850 712 50M	3.464 101 62M
$(E/m)_{\mathrm{circle}}$	0.976 187 060	0.942 809 042
$v_{\mathrm{x,shell,circle}}$	0.235 702 260	0.5
$(L/m)_{\mathrm{transfer}}$	3.787 166 42M	3.787 166 42M
$(E/m)_{\mathrm{transfer}}$	0.965 541 773	0.965 541 773
$v_{\mathrm{x,shell,transfer}}$	0.186 052 102	0.266 880 257

NOTE: All shell velocities in this table are tangential, in the positive shell x-direction.

Long numbers
in tables

86   **Comment 1. Significant digits**
87   In this chapter we analyze several unstable (knife-edge) circular orbits.
88   Interactive software, such as GRorbits, requires accurate inputs to display the
89   orbit of an unpowered probe that stays in an unstable circular orbit for more than
90   one revolution. To avoid clutter, we put numbers with many significant digits into
91   tables.

92   **Comment 2. Long subscripts**
93   In Table 1 the symbols $v_{\mathrm{x,shell,insert}}$, $v_{\mathrm{x,shell,circle}}$, and $v_{\mathrm{x,shell,transfer}}$ have long,
94   ungainly subscripts. We need long subscripts to fully describe these velocity
95   components: that they are x-components measured in a local shell frame and
96   whether they describe insertion speed into a circular orbit, speed in that circular
97   orbit, or transfer between circular orbits.

Impulse
rocket thrusts

98   **Comment 3. Impulse rocket thrusts**
99   We assume that each change in vehicle speed results from a quick rocket thrust,
100   an impulse. In practice there is no hurry; some efficient rocket engines provide
101   low thrust, which carries the vehicle through a series of intermediate orbits. To
102   analyze the outcome of a slow burn complicates calculations and does not add
103   to our understanding. So our vehicles use quick rocket thrusts to transfer from
104   one orbit to another.

105   **Comment 4. Which direction is the "rocket thrust"?**
106   What is the meaning of the phrase *outward rocket thrust*? The rocket fires in one
107   direction; the probe or spaceship that carries the rocket changes speed in the
108   opposite direction. We define *outward rocket thrust* to mean that the rocket burn
109   tends to move the rocket to larger r. Similarly, the *inward rocket thrust* tends to
110   move the rocket to smaller r.

Insert into
circular orbit

111   When the spaceship moves tangentially for an instant at $r = 20M$, the
112   spaceship fires a tangential rocket thrust to put it into the stable circular orbit
113   at that r. What change in tangential velocity must this rocket thrust provide?
114   Tangential velocity in *which* frame? Our policy: make every measurement in a
115   local inertial frame; for that purpose, choose the local *shell* frame. Box 2 in
116   Section 7.4 gives shell frame coordinates from which we derive shell
117   components of velocity. For reasons that will become apparent, we start with
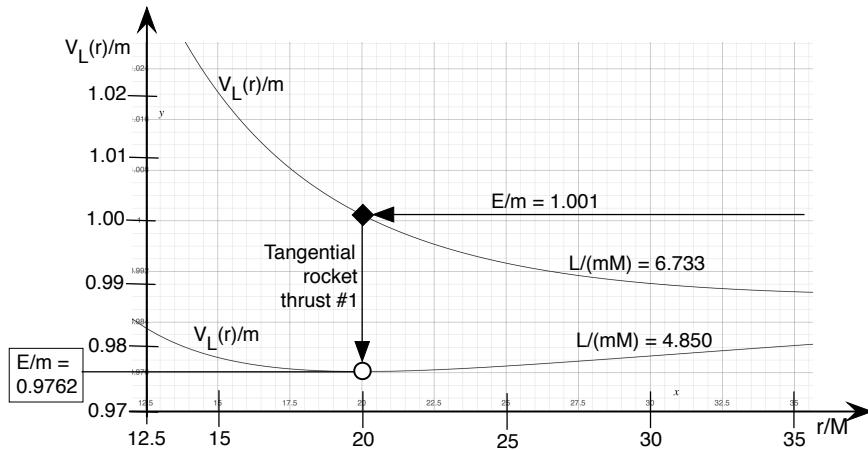
FIGURE 3 At the instant when the incoming spaceship moves tangentially at the radial turning point $r = 20M$ (Figure 2), it fires tangential rocket thrust #1 that changes its map energy and map angular momentum to insert it into a stable circular orbit.

definitions of $dt_{\text{shell}}/d\tau$, $dy_{\text{shell}}/d\tau$, and $dx_{\text{shell}}/d\tau$, each with wristwatch time differential $d\tau$ in the denominator.

$$\frac{dt_{\text{shell}}}{d\tau} = \lim_{\Delta\tau \to 0} \frac{\Delta t_{\text{shell}}}{\Delta\tau} \tag{12}$$

$$= \left(1 - \frac{2M}{r}\right)^{-1/2} \left[\left(1 - \frac{2M}{r}\right)\frac{dT}{d\tau} - \left(\frac{2M}{r}\right)^{1/2}\frac{dr}{d\tau}\right] \tag{13}$$

$$= \left(1 - \frac{2M}{r}\right)^{-1/2} \frac{E}{m} \tag{14}$$

The last step uses equation (1). Similarly:

$$\frac{dy_{\text{shell}}}{d\tau} = \lim_{\Delta\tau \to 0} \frac{\Delta y_{\text{shell}}}{\Delta\tau} = \left(1 - \frac{2M}{r}\right)^{-1/2} \frac{dr}{d\tau} \tag{15}$$

To find an expression for $dr/d\tau$ in this equation, combine equations (4) and (5):

$$\frac{dr}{d\tau} = \pm\left[\left(\frac{E}{m}\right)^2 - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{L^2}{m^2 r^2}\right)\right]^{1/2} \tag{16}$$

And finally:

$$\frac{dx_{\text{shell}}}{d\tau} = \lim_{\Delta\tau \to 0} \frac{\Delta x_{\text{shell}}}{\Delta\tau} = r\frac{d\phi}{d\tau} = \frac{L}{mr} \tag{17}$$

Shell velocity
components

The last step uses equation (2). To complete the derivation of shell velocity components, note, for example, that $v_{\text{y,shell}} = (dy_{\text{shell}}/d\tau)(d\tau/dt_{\text{shell}})$, so from (15) and (14):

$$v_{\mathrm{y,shell}} = \frac{dr/d\tau}{E/m} = \pm \left[1 - \left(\frac{E}{m} \right)^{-2} \left(1 - \frac{2M}{r} \right) \left(1 + \frac{L^2}{m^2 r^2} \right) \right]^{1/2} \quad (18)$$

$$v_{\mathrm{x,shell}} = \left(1 - \frac{2M}{r} \right)^{1/2} \frac{L}{rE} \quad\quad\quad (19)$$

127

128   Use the first two entries in Table 1 plus equation (19) to calculate the
129 value of $v_{\mathrm{x,shell,insert}}$ at $r = 20M$ (where the shell y-component
130 $v_{\mathrm{y,shell,insert}} = 0$) and check the result in the third line of Table 1.

131

QUERY 1. Tangential shell velocity in a circular orbit

A. What is the tangential shell velocity of the spaceship in the circular orbit at r? Combine
equations (6) and (8) to find an expression for L/E and substitute the result into (19):

$$v_{\mathrm{shell,circle}} = \left(\frac{M}{r} \right)^{1/2} \left(1 - \frac{2M}{r} \right)^{-1/2} \quad\quad (\text{circular orbit}, r > 3M) \quad\quad (20)$$

135

B. Show that your derivation is not valid unless $r > 3M$.
C. Use (20) to calculate a value for $v_{\mathrm{shell,circle}}$ at $r = 20M$. Check your answer with the entry in
Table 1.   138

139

140   Table 1 tells us that the shell frame velocity $v_{\mathrm{x,shell,insert}}$ of the spaceship
141 in its insertion orbit is greater than its shell frame velocity $v_{\mathrm{x,shell,circle}}$ in the
142 circular orbit. Therefore a rocket thrust must bring the spaceship's shell
143 velocity down to that of the circular orbit.

Use velocity
addition laws.

144   Einstein shouts, "Look out! To calculate the needed change in spaceship
145 velocity to be provided by the rocket thrust, you do *not* use the difference
146 between $v_{\mathrm{x,shell,insert}}$ and $v_{\mathrm{x,shell,circle}}$." Why not? Because in special relativity
147 (which rules in every local inertial frame), velocities do not simply add or
148 subtract.

149   In what local inertial frame can we measure directly the change in velocity
150 provided by the rocket thrust? That would be the local inertial frame in which
151 the spaceship is initially at rest just before the thrust. Just before the rocket
152 thrust, the spaceship moves at velocity $v_{\mathrm{x,shell,insert}}$ in the shell frame. We call
153 the local inertial frame in which the spaceship is at rest the **instantaneous**
154 **initial rest frame** or IIRF.

155   **DEFINITION 2. Instantaneous Initial Rest Frame (IIRF)**

Definition:
Instanteous Initial
Rest Frame (IIRF)

156   The instantaneous initial rest frame (IIRF) is the local inertial frame in
157   which a rocket is at rest just before it fires a rocket thrust to change its
158   velocity with respect to that frame. We use the subscript IIRF to indicate

TABLE 9.2 Rocket Thrusts in Instantaneous Initial Rest Frames (IIRF)

Thrust	at $r =$	Δv_{IIRF} component	Description
#1	$20M$	$\Delta v_{\text{x,IIRF1}} = -0.090\ 132\ 846\ 2$	into circular orbit
#2	$20M$	$\Delta v_{\text{x,IIRF2}} = -0.051\ 927\ 321\ 7$	into transfer orbit
#3	$6M$	$\Delta v_{\text{x,IIRF3}} = -0.269\ 017\ 469$	into ISCO
#4	$6M$	$\Delta v_{\text{x,IIRF4}} = 0.060\ 908\ 153\ 8$	into transfer orbit
#4	$6M$	$\Delta v_{\text{y,IIRF4}} = -\ 0.228\ 989\ 795$	into transfer orbit

NOTE: After thrust #4, the probe coasts into the unstable circular orbit at $r = 4M$.

159 quantities in this rest frame, as in the symbols $\Delta v_{\text{x,IIRF}}$ and $\Delta v_{\text{y,IIRF}}$
160 for the change in velocity components in the IIRF frame caused by that
161 rocket impulse. We describe four different IIRF thrusts, listed with an
162 additional number 1 through 4 added to the subscript (Table 2).

163 Special relativity addition of velocities gives us our first, tangential, IIRF
164 rocket-thrust change $\Delta v_{\text{x,IIRF1}}$ with the number 1 added to the subscript. This

IIRF1 transfer
velocity change

165 rocket thrust must reduce the shell speed of the spaceship. From equation (54)
166 of Section 1.13,

$$\Delta v_{\text{x,IIRF1}} = \frac{v_{\text{x,shell,circle}} - v_{\text{x,shell,insert}}}{1 - v_{\text{x,shell,insert}} v_{\text{x,shell,circle}}} \tag{21}$$

$$= -0.090\ 132\ 846\ 2 \qquad \text{(into circular orbit at } r = 20M\text{)}$$

167 Put this numerical value into Table 2. This rocket-thrust velocity change (-27
168 021 kilometers/second) inserts the incoming spaceship into the circular orbit
169 at $r = 20M$.

?

170 **Objection 1.** *Wait! The two velocities, $v_{\text{x,shell,circle}}$ and $v_{\text{x,shell,insert}}$ are*
171 *measured in the same local inertial shell frame. The difference in*
172 *x-components is the measured difference in x-components; why confuse*
173 *things with complicated equation (21)?*

!

174 Remember in special relativity the *law of addition of velocities* between two
175 inertial frames in relative motion (Part A of Exercise 17 , Section 1.13)?
176 Equation (21) could be called the *law of subtraction of velocities*—Part B of
177 that earlier exercise. The complication of equation (21) does not require
178 general relativity.

?

179 **Objection 2.** *Wow, that is quite a long vertical line in Figure 3. How fast*
180 *does the probe move along that line? That quick transition must violate the*
181 *light-speed limit!*

!

182 No, the probe does not change any global coordinate, T, r, or ϕ, as it
183 traverses the (idealized) vertical line. That transition results from a rocket
184 thrust; it simply changes L and E almost instantaneously (Comment 3).
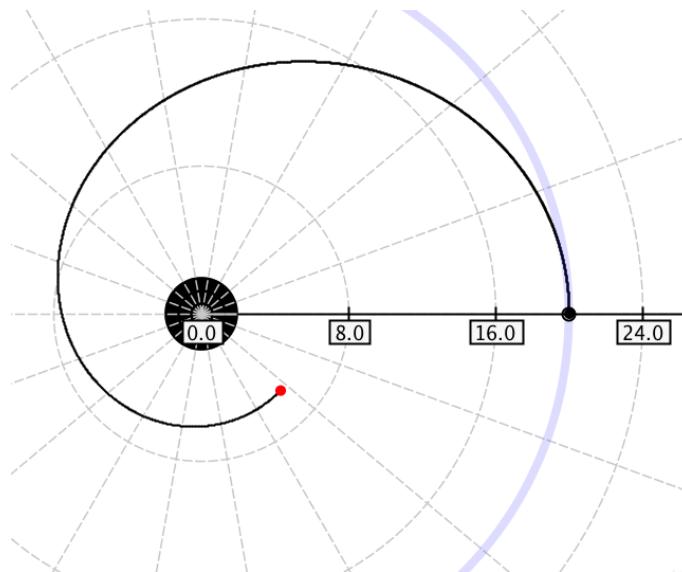
FIGURE 4   Transfer orbit in which the unpowered probe coasts from tangential motion at $r_\mathrm{A} = 20M$ to tangential motion at $r_\mathrm{ISCO} = 6M$. Figure 5 shows the effective potential for this transfer and change in tangential speed required to put the probe into this transfer orbit.

?

185   **Objection 3.** *Your analysis of insertion into a circular orbit takes no*
186   *account of mass loss due to required rocket thrusts. Whenever spaceship*
187   *mass changes, its map energy and map angular momentum also change.*

!

188   Right you are. However, constants of motion in our equations are map
189   energy and map angular momentum *per unit mass.* Map energy E/m and
190   map angular momentum $L/(mM)$ are unitless. Therefore the initial mass
191   of the spaceship (before a rocket thrust) and the final spaceship mass
192   (after the rocket thrust) do not affect these equations.

9.3 ■ TRANSFER TO THE ISCO

194   *Get closer*

195   The spaceship completes observations from the stable circular orbit at
196   $r = 20M$ and its captain wants to make further observations from a smaller
197   circular orbit—still outside the event horizon. To take the entire spaceship to
198   this smaller orbit requires a large amount of rocket fuel; instead the captain
199   launches a small probe toward the smaller orbit.
200   What r-value shall we choose for the inner circular orbit? Be bold! Take

Transfer to circular   201   the probe all the way down to the so-called Innermost Stable Circular Orbit at
orbit at $r_\mathrm{ISCO} = 6M$.   202 $r_\mathrm{ISCO} = 6M$ (Definition 6, Section 8.5).
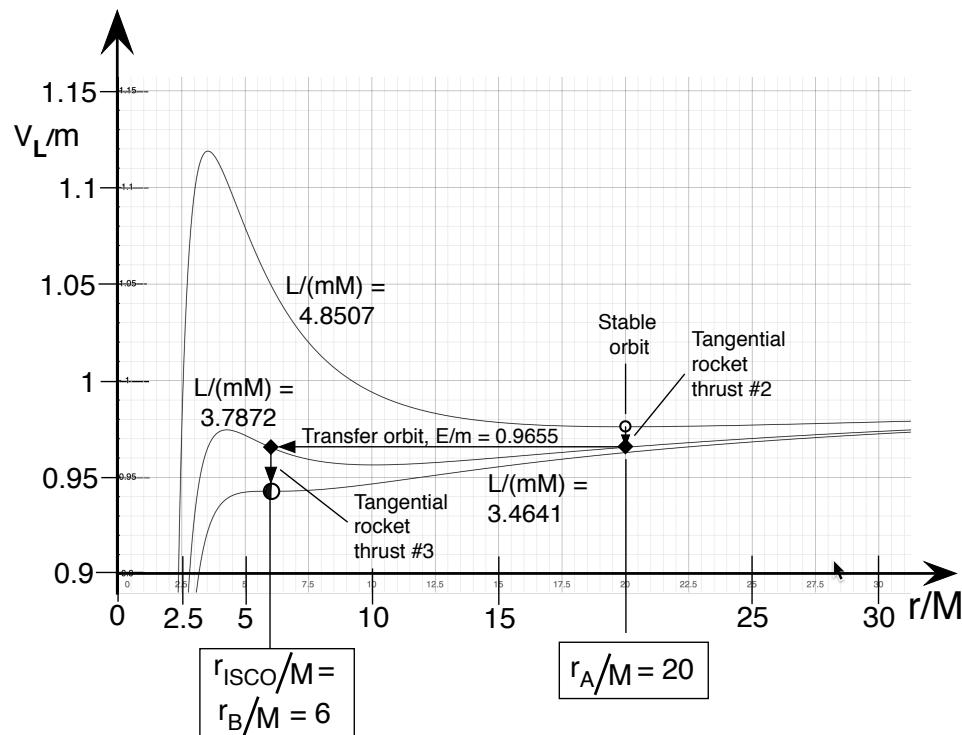
FIGURE 5   Transfer orbit between sequential tangential rocket thrusts #2 and #3. This maneuver moves the probe from the stable circular orbit at $r = 20M$ to the half-stable ISCO at $r_{\rm ISCO} = 6M$. Figure 4 plots this transfer orbit on the $[r, \phi]$ slice.

203   **Comment 5. ISCO as a limiting case**
204   The ISCO is hazardous because it's "half stable" and may lead to a death spiral
205   inward through the event horizon. To prevent this, the inner circular orbit r-value
206   should be slightly greater than $r_{\rm ISCO}$ to make it fully stable. In what follows we
207   ignore this necessary small r-adjustment.

208   Figure 4 shows a transfer orbit, tangential at both $r_{\rm A} = 20M$ and
209    $r_{\rm B} = r_{\rm ISCO} = 6M$. Recall that these radii are called **radial turning points,**
210   because at both r-values $dr/d\tau = 0$, so the orbiter instantaneously sweeps
211   around only tangentially. Figure 5 displays the corresponding map energy on
212   the effective potential plot.

213

QUERY 2. Profile of transfer orbit
In 1925 Walter Hohmann described a transfer orbit between two planetary orbits around our Sun as
"half an ellipse." Half an ellipse would have maxima of $r_{\rm A}$ and $r_{\rm B}$ on opposite sides of the center of
attraction. The orbit plot in Figure 4 does not look like half an ellipse. Why is this different from
Hohmann's prediction?

219

220   We seek a transfer orbit between the specified Above circular orbit at
221 r_A/M and the Below circular orbit at r_B/M; Figure 5 shows this transfer. In
222 equation (4), $dr/d\tau = 0$ at the two turning points r_A/M and r_B/M, which
223 yields:

$$\left(\frac{E}{m}\right)^2 = \left(\frac{V_L(r_A)}{m}\right)^2 = \left(\frac{V_L(r_B)}{m}\right)^2 \qquad \text{(at turning points)} \qquad (22)$$

224   Look first at the right equality in (22), in which the square of the effective
225 potential (5) has the same value at two different r. Write down this equality

Transfer orbit   226 and solve the resulting equation for $(L/m)^2$. The result is equation (23). Next
map L and E   227 look at the left equality in (22), in which the square of the map energy
228 $(E/m)^2$ is equal to the square of the effective potential at either r. Write down
229 this equality and solve the resulting equation for $(E/m)^2$. The result is
230 equation (24).

$$\left(\frac{L}{m}\right)^2_{\text{transfer}} = \frac{2Mr_A^2 r_B^2(r_A - r_B)}{r_A^3(r_B - 2M) - r_B^3(r_A - 2M)} \quad \text{(between circular orbits)} \ (23)$$

$$\left(\frac{E}{m}\right)^2_{\text{transfer}} = \frac{(r_A - 2M)(r_B - 2M)(r_A^2 - r_B^2)}{r_A^3(r_B - 2M) - r_B^3(r_A - 2M)} \quad \text{(between circular orbits)} \ (24)$$

231

232

QUERY 3. Transfer either way
Show that equations (23) and (24) are both symmetrical in r_A and r_B. In other words, show that the
same values of $(L/m)_{\text{transfer}}$ and $(E/m)_{\text{transfer}}$ apply, irrespective of the direction of transfer between
the circular orbits. Is this result obvious?

237

238   Substitute values $r_A = 20M$ and $r_B = r_{\text{ISCO}} = 6M$ into equations (23) and
239 (24); enter resulting values of L/m and E/m into Table 1. Then equations (18)
240 and (20) give us values of $v_{\text{x,shell,transfer}}$ and $v_{\text{x,shell,circle}}$. These results allow us
241 to compute the rocket thrust needed to put the probe into the transfer orbit.
242 This is our second, also tangential, instantaneous initial rest frame IIRF thrust
IIRF2 transfer   243 (Definition 2) with the number 2 added to the subscript, $\Delta v_{\text{x,IIRF2}}$.
velocity change

$$\Delta v_{\text{x,IIRF2}} = \frac{v_{\text{x,shell,transfer}} - v_{\text{x,shell,circle}}}{1 - v_{\text{x,shell,transfer}} v_{\text{x,shell,circle}}} \qquad \text{(into transfer orbit} \qquad (25)$$

$$= -0.051\ 927\ 321\ 7 \qquad\qquad \text{from } r = 20M \text{ to } r_{\text{ISCO}})$$

244   Enter this numerical result into Table 2. This rocket-thrust velocity change
245 ($-15\ 567$ kilometers/second) inserts the probe into a transition orbit that
246 carries it from tangential motion at $r = 20M$ down to tangential motion at
247 $r_{\text{ISCO}} = 6M$.

?

248 **Objection 4.** *You talk about moving into a circular orbit and transferring*
249 *between orbits. But what will our orbiting observers **see**? You have told us*
250 *nothing about what they see as they look around.*

!

251 Guilty as charged! Section 7.7 showed only what a raindrop diver sees
252 radially inward and radially outward as she plunges to the center of the
253 black hole. Beyond that, we have made no predictions whatsoever about
254 what any observer sees. For example: In what local frame direction must
255 an observer look to see a particular star? What must we know to make
256 such predictions? Chapters 13 answers these questions. The cosmic trip
257 planner must read beyond the present chapter!

258   When the probe reaches $r_{\text{ISCO}} = 6M$, it travels tangentially for an instant
259 at shell velocity $v_{\text{x,shell,transfer}}$. Then a third insertion rocket thrust changes

IIRF3 transfer
velocity change

260 this shell velocity to $v_{\text{x,shell,circle}}$ for the circular orbit at r_{ISCO}. Table 1 has
261 values of both of these velocities. What insertion rocket thrust does this? As
262 before, it is a tangential thrust in the instantaneous inertial rocket frame IIRF
263 (Definition 2), with the number 3 added to the subscript, $\Delta v_{\text{x,IIRF3}}$.

$$\Delta v_{\text{x,IIRF3}} = \frac{v_{\text{x,shell,transfer}} - v_{\text{x,shell,circle}}}{1 - v_{\text{x,shell,transfer}} v_{\text{x,shell,circle}}} \tag{26}$$

$$= -0.269\ 017\ 469 \qquad \text{(into circular orbit at } r_{\text{ISCO}} = 6M)$$

264 Enter the numerical result in Table 2. This rocket-thrust velocity change
265 ($-86\ 494$ kilometers/second) inserts the probe into the circular orbit at
266 $r_{\text{ISCO}} = 6M$.

9.4 ■ TRANSFER TO AN UNSTABLE CIRCULAR ORBIT

268 *Put the probe at risk!*

269 Thus far we have inserted our spaceship into a stable circular orbit at
270 $r = 20M$, then transferred a probe down to the half-stable circular orbit at

Transfer to
unstable orbit
at $r = 4M$

271 $r_{\text{ISCO}} = 6M$. Now the spaceship captain wants to make observations even
272 closer to the black hole. She decides to transfer the probe from $r_{\text{ISCO}} = 6M$ to
273 the unstable circular orbit at $r = 4M$, a maneuver shown in Figures 6 and 7.

274

QUERY 4. Unstable circular orbit at $r = 4M$

A. Show that the unstable circular orbit at $r = 4M$ has map angular momentum $L/m = 4M$.

B. Show that the unstable circular orbit at $r = 4M$ has map energy $E/m = 1$.

C. Make an argument that the transfer orbit from $r = 6M$ to $r = 4M$ in Figures 6 and 7 must have the same values of map energy and map angular momentum given in the first two items of this Query.
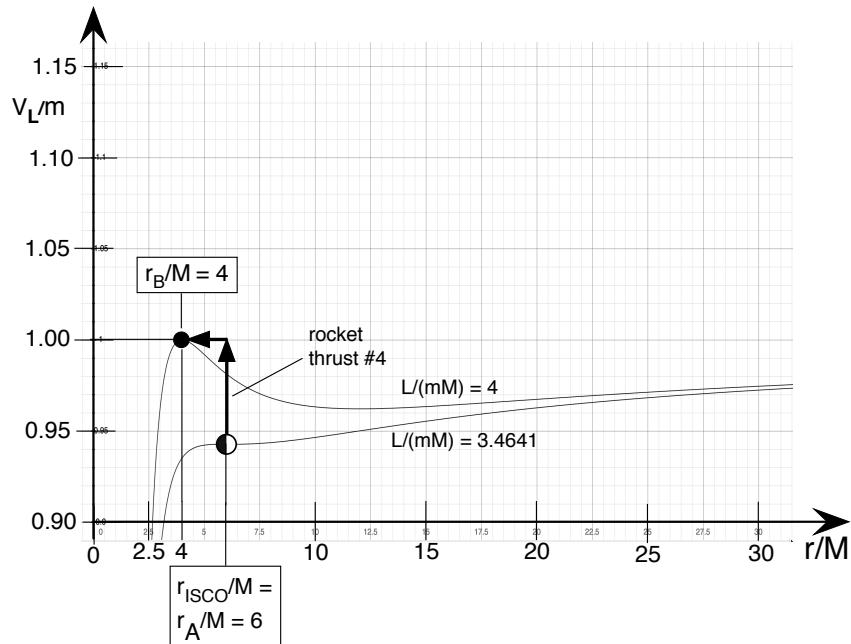
280

FIGURE 6   Probe transfer orbit between half-stable orbit at $r_{\text{ISCO}} = 6M$ and unstable circular orbit at $r = 4M$. See Figure 7.

D. Verify the bottom right hand entry in Table 3, namely that at $r = 4M$,

$$v_{\text{x,shell,circle}} = v_{\text{x,shell,transfer}} = |v_{\text{shell,transfer}}|$$

Transfer orbits have radial turning points where $E/m = V_{\text{L}}(r)$. Usually these turning points are not at an extremum of the effective potential, so they are not at r-values of circular orbits. In this case, however, we need a rocket thrust to *create* the extremum for a circular orbit at that r-value.

At a maximum of the effective potential, the turning point occurs at the r-value of the circular orbit, so we need no rocket thrust to put the probe into that circular orbit. Figure 6 shows this special case: The probe moves to smaller r along the horizontal arrow in Figure 6. As it does so it reaches the effective potential maximum at $r = 4M$ where it automatically enters the unstable circular orbit at that r-value. So we need only a single rocket thrust at $r = 6M$ to change map energy and map angular momentum to that of the circular orbit at $r = 4M$ (Figure 7).

No rocket thrust needed for insertion into unstable orbit.

?

Objection 5. *Once the rocket thrust #4 shoots the probe upward in Figure 6 to map energy $E/m = 1$, why should the probe go left in that figure, to smaller r? Why doesn't it go right, to larger r?*
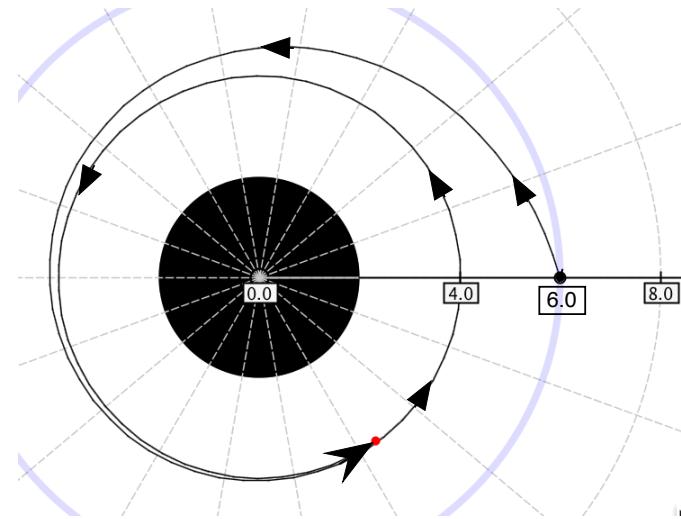
FIGURE 7   Transfer orbit from $r_{\mathrm{ISCO}} = 6M$ to the unstable circular orbit at $r = 4M$ (Figure 6). This requires a velocity $v_{\mathrm{shell,transfer}}$ inward from $90°$ by 19.471 degrees, with shell velocity components and magnitude given in Table 3.

!

299   Figure 7 and Table 3 show the answer: The rocket thrust is not tangential
300   but has an inward r-component.

301   Query 4 already tells us the map values $E/m = 1$ and $L/m = 4M$ of the
302   leftward horizontal arrow in Figure 6. Because the rocket thrust is not

Need two thrust components for transfer orbit

303   tangential, we need to apply the full set of equations (18) and (19) to find the
304   shell components of the velocity in the transfer orbit. Enter these results for
305   $v_{\mathrm{y,shell,transfer}}$ and $v_{\mathrm{x,shell,transfer}}$ in Table 3.
306   To start this transfer from r_{ISCO} we use the fourth rocket thrust measured
307   in the instantaneous initial rest frame. This thrust requires two components,
308   which we call $\Delta v_{\mathrm{x,IIRF4}}$ and $\Delta v_{\mathrm{y,IIRF4}}$, with the number 4 added to the
309   subscript. In this case we must adapt both velocity addition equations (54) in
310   Section 1.13.

$$\Delta v_{\mathrm{x,IIRF4}} = \frac{v_{\mathrm{x,shell,transfer}} - v_{\mathrm{x,shell,circle}}}{1 - v_{\mathrm{x,shell,circle}} v_{\mathrm{x,shell,transfer}}} \quad \text{(into the transfer orbit...} \quad (27)$$

$$\Delta v_{\mathrm{y,IIRF4}} = \frac{v_{\mathrm{y,shell,transfer}}}{\gamma_{\mathrm{x,shell,circle}}\left(1 - v_{\mathrm{x,shell,circle}} v_{\mathrm{x,shell,transfer}}\right)} \quad \text{...from } r = 6M \ (28)$$

$$\text{where} \quad \gamma_{\mathrm{x,shell,circle}} = \left(1 - v_{\mathrm{x,shell,circle}}^2\right)^{-1/2} \quad \text{...to } r = 4M) \quad (29)$$

311   Substitute into these equations from $r = r_{\mathrm{ISCO}} = 6M$ values in Tables 1
312   and 3 and enter the resulting components into Table 2. This rocket thrust,
313   which corresponds to the vertical arrow in Figure 6, causes a velocity change
314   of magnitude, $|\Delta v_{\mathrm{IIRF4}}| = 0.236\ 951\ 745 = 71\ 036$ kilometers/second.

TABLE 9.3   Numerical values for transfer from $r_{\text{ISCO}} = 6M$ to $r = 4M$

Values of	$r_{\text{ISCO}} = 6M$	$r = 4M$
$(L/m)_{\text{transfer}}$	$4M$	$4M$
$(E/m)_{\text{transfer}}$	1	1
$v_{\text{x,shell,transfer}}$	0.544 331 054	0.707 106 781
$v_{\text{y,shell,transfer}}$	$-0.192\ 450\ 090$	0
$|v_{\text{shell,transfer}}|$	0.577 350 269	0.707 106 781
$\theta_{\text{x,shell}}$	$-19.471\ 220\ 6°$	0
$v_{\text{x,shell,circle}}$	0.500 000 000	0.707 106 781

Good-bye probe!

315   Our probe coasts to the unstable circular orbit at $r = 4M$, an effective
316   potential peak close to the black hole. After it completes measurements there,
317   the captain decides to dispose of the probe. To do this, she commands the
318   probe to fire a tiny inward rocket thrust to tip it off the effective potential
319   peak and send it spiraling inward across the event horizon. Good job!
320   Section 9.5 applies some of what we have learned to analyze Larry Niven's
321   short story "Neutron Star."

9.5 ■ "NEUTRON STAR" BY LARRY NIVEN

323   *Close to a neutron star? Look out!*

324   Larry Niven's science fiction short story "Neutron Star" describes the trip by
325   spaceship pilot Beowulf Schaeffer to discover why two earlier pilots died while
326   orbiting a neutron star. Sponsors of Beowulf's trip are aliens called

Why did earlier
explorers die?

327   puppeteers, who manufacture spaceship hulls that are utterly indestructable
328   and—so they claim—impenetrable. Naturally, the death of two pilots in an
329   "impenetrable" puppeteer spaceship hull has reduced sales. The puppeteers
330   want to know what deadly force has managed to enter their high-tech hulls.
331   As Beowulf approaches the neutron star, the long axis of his spaceship
332   inexorably orients along a radial line to the star (Why?). Beowulf suddenly
333   realizes that he must position himself at the point in the spaceship where at

Passage through
closest approach

334   least one part of his body feels no gravity in order to be in free-fall motion
335   around the neutron star. Here is Niven's description of his passage through the
336   r-coordinate of closest approach:

337   *My time was up. A red disk leapt up at me; the ship swung*
338   *around me; I gasped and shut my eyes tight. Giants' hands*

"Giants' hands
gripped . . ."

339   *gripped my arms and legs and head, gently but with great*
340   *firmness, and tried to pull me in two. In that moment it came*
341   *to me that Peter Laskin had died like this. He'd made the*
342   *same guesses I had, and he'd tried to hide in the access tube.*
343   *But he'd slipped . . . as I was slipping . . . From the control*
344   *room came a multiple shriek of tearing metal. I tried to dig my*
345   *feet into the hard tube walls. Somehow they held.*

Close-call
survival

346   According to Niven's story, Beowulf is (barely!) able to cling to the point
347   of zero local gravity, though the skin on his extremities is injured. After
348   returning to base, he reports to the puppeteers that the deaths of earlier
349   explorers were due to their slipping from this gravity zero point and falling to
350   the front (or back) of the spaceship.

?

351   **Objection 6.** *What in (or out of) this world is happening to Beowulf? His*
352   *orbit around the neutron star is similar to those we use to insert our*
353   *spaceship into a circular orbit. Why is Beowulf in danger, and why did*
354   *earlier explorers die?*

!

355   "All politics is local," said politician Tip O'Neill. A monster may lurk at
356   opposite ends of your spaceship. In "Neutron Star" the monster is *tidal*
357   *acceleration*, which can be lethal.

358   Tidal acceleration is nothing new for us. Section 7.9 introduced it for the
359   radial fall into the black hole, and in the present chapter Section 9.7,
Killer tides
360   Appendix: Killer Tides, gives expressions for radial and tangential tidal
361   accelerations. This information allows us to answer the question, "Can
362   Beowulf Schaeffer survive his transit past the neutron star?"
363   We need numerical values from "Neutron Star" in order to apply tidal
Survival?
364   acceleration expressions from Section 9.7. Larry Niven tells us that (a) the
365   neutron star's mass is 1.3 times the mass of our Sun, (b) the minimum
366   r-coordinate of approach is approximately 10.5 kilometers, so that
367   $r_{\min} \approx 5.5M$. (The neutron star is also spinning, but too slowly to have a
368   significant effect on Beowulf's global orbit or local safety.)

369

QUERY 5. Einstein predicts Beowulf Schaeffer's fate
Use the parameters in the preceding paragraph to find out whether or not Beowulf Schaeffer survives
tidal accelerations during his encounter with the neutron star. Assume that the distant speed of
approach to the neutron star is nonrelativistic, so that $E/m \approx 1$.

A. Use (3) to determine v_{shell} at the closest approach $r_{\min}$.
B. By what multiple is the radial tidal effect (in the local spaceship Δy_{ship} direction) larger than
   the Newtonian prediction?
C. At the moment of closest approach to the neutron star, Beowulf Schaeffer extends his arm one
   meter radially inward. What happens to him next?
D. Give a definitive answer to the question, "Can Beowulf Schaeffer survive the trip described in
   "Neutron Star"? (When our class sent numerical results to Larry Niven, he replied, "Thank you
   for the calculations. I'm not sure how I will use them, but thanks anyway.")
E. If you conclude that Beowulf cannot survive the "Neutron Star" trip, find an r-coordinate of
   closest approach to the neutron star at which Beowulf Schaeffer can survive. State your criteria
   for *survival*. On the way to this result, give a specific numerical value for $\Delta g/\Delta y_{\text{ship}}$ that, in
   your estimate, is survivable.

QUERY 6. Blackmail
Discussion question: Beowulf Schaefer blackmails the secretive puppeteers by threatening to reveal that they come from a moonless world. How does he know that?

QUERY 7. *Optional:* **Swimming in spacetime?**
A massive mother ship is in a circular orbit with its long dimension tangential with respect to the black hole. Astronauts inside extend a mechanical arm radially inward toward the black hole. The "hand" on this arm experiences a radially inward force.

   A. Can such a maneuver be used to change the orbit of the mother ship?
   B. Can similar maneuvers provide a method for balancing a spaceship in a circular knife-edge orbit without using rockets?
   C. Using repeated "calisthenics," can a freely-floating astronaut "swim" around the mother ship? (See "Swimming in Spacetime" in the references.)
   D. Do such maneuvers violate the laws of conservation of map energy or map angular momentum?
   E. Do similar maneuvers work in flat spacetime?

9.6 ■ A COMFORTABLE CIRCULAR ORBIT

How close to the black hole?

Meaning of "comfortable"?

Up to this point, our description of circular orbits has a serious flaw: We do not answer the question, "What is the minimum r-value of a circular orbit in which the astronaut will be comfortable?" Our answer to this question has three parts:

   • Part I. What are the tidal accelerations in a circular orbit of given r-coordinate? To answer this question, we consult Section 9.7, Appendix: Killer Tides.

   • Part II. What is the maximum tidal acceleration for which a human is comfortable?

   • Part III. What is the minimum r-coordinate of a circular orbit (Part I) for which a human is comfortable (Part II)?

Instead of choosing an orbit that is comfortable for a human, we can replace the human with a probe hardened to withstand hundreds or thousands of times the tidal accelerations that would injure or kill a person.

Part I: Tidal acceleration in circular orbit

In order to apply tidal equations (46) through (48) to a circular orbit, we need the square of the tangential shell velocity in (10).

Think of an astronaut in a circular orbit with the long axis of his body oriented along the radial direction. His height is larger than his width, so we carry out our calculations for the radial tidal component only, knowing that the other components will be smaller. Half his height provides a value for Δy_{local} in equation (46). Substitute (10) into (46) and rearrange so the right side of the equation contains only expressions in r.

Tidal acceleration in circular orbit

$$\Delta g_{\text{local},y} \approx \frac{M}{\bar{r}^3} \left(\frac{2\bar{r} - 3M}{\bar{r} - 3M} \right) \Delta y_{\text{local}} \qquad \text{(circular orbit)} \qquad (30)$$

Part II: Define human comfort.

How large a tidal acceleration is comfortable for a human being? The answer is different for people of different heights. Here we treat our human astronaut gently, using the definition employed in Section 7.9 under the assumption that he is oriented along a radial line, with head above feet. Then with his stomach in free fall, the astronaut remains comfortable if his head is accelerated upward with the acceleration it would experience on Earth—call it g_{E}—and his feet are accelerated downward with the same magnitude of Earth acceleration.

Tidal acceleration for human comfort

Assume the astronaut is approximately two meters tall, so his measured distance between head and stomach is one meter, the same as the separation between stomach and feet. Then $\Delta y_{\text{local}} = 1$ meter in equation (30).

Part III: Minimum-r circular orbit for human comfort

The acceleration g_{E} at Earth's surface has the numerical value $g_{\text{E}} = 1.09 \times 10^{-16}$ meter^{-1} (inside the front cover). We want to insert g_{E} into (30) when the circling astronaut's "half height" is $\Delta y_{\text{local}} = 1$ meter:

Minimum r for comfort?

$$g_{\text{E}} = \Delta g_{\text{local},y} \approx \frac{M}{\bar{r}_{\text{comfort}}^3} \left(\frac{2\bar{r}_{\text{comfort}} - 3M}{\bar{r}_{\text{comfort}} - 3M} \right) \times 1\,\text{meter} \quad \text{(human comfort limit)} \qquad (31)$$

$$g_{\text{E}} \approx \frac{M^{-2}}{(\bar{r}_{\text{comfort}}/M)^3} \left(\frac{2\bar{r}_{\text{comfort}}/M - 3}{\bar{r}_{\text{comfort}}/M - 3} \right) \times 1\,\text{meter} \qquad (32)$$

In this equation, $\bar{r}_{\text{comfort}}$ refers to the smallest r-value of the circular orbit in which the observer is comfortable. Multiply the left and right sides of (32) by M^2 and divide by g_{E}. The result is

$$M^2 \approx \frac{1}{(\bar{r}_{\text{comfort}}/M)^3} \left(\frac{2\bar{r}_{\text{comfort}}/M - 3}{\bar{r}_{\text{comfort}}/M - 3} \right) \frac{1\,\text{meter}}{g_{\text{E}}} \quad \text{(human comfort limit)} \qquad (33)$$

We can rearrange (33) to give the mass of the black hole in number of Suns, M/M_{Sun}, as a function of the minimum r-value, r_{comfort}, of the circular orbit in which a human astronaut will be comfortable:

FIGURE 8 The horizontal axis, r_{comfort}/M, gives the minimum-r circular orbit in which a human will be comfortable. On the vertical axis, M/M_{Sun} is a number equal to the mass of the black hole in units of the mass of our Sun. Arrows and little filled circles illustrate solutions of Sample Problems 1A through 1D.

$$\frac{M}{M_{\text{Sun}}} = \frac{1}{M_{\text{Sun}}} \left(\frac{1\,\text{meter}}{g_{\text{E}}} \right)^{1/2} \left[\frac{1}{\left(\bar{r}_{\text{comfort}}/M \right)^3} \left(\frac{2\bar{r}_{\text{comfort}}/M - 3}{\bar{r}_{\text{comfort}}/M - 3} \right) \right]^{1/2} \quad (34)$$

$$= 6.47 \times 10^4 \left[\frac{1}{\left(\bar{r}_{\text{comfort}}/M \right)^3} \left(\frac{2\bar{r}_{\text{comfort}}/M - 3}{\bar{r}_{\text{comfort}}/M - 3} \right) \right]^{1/2} \quad (35)$$

(minimum-r circular orbit for human comfort)

451

452   The last step substitutes values of M_{Sun} and g_{E} from inside the front cover.
453   Verify that both sides of this equation are unitless. Figure 8 plots the curve of
454   this equation. Sample Problems 1 explain the arrows.

Sample Problems 1. Minimum-r Circular Orbit for Human Comfort

PROBLEM 1A

What is the numerical value of M/M_{Sun} for which $r_{\mathrm{comfort}}/M = 20$ is the minimum circular orbit in which a human feels comfortable? What is the value of r_{comfort} in meters?

SOLUTION 1A

Figure 8 shows that at $r_{\mathrm{comfort}}/M = 20$, $M/M_{\mathrm{Sun}} \approx 10^3$, indicated by point A in the figure. The value of r_{comfort} in meters is $r_{\mathrm{comfort}} = 20 \times M$ meters $= 20 \times (M/M_{\mathrm{Sun}}) \times M_{\mathrm{Sun}}$ meters $\approx 20 \times 10^3 \times 1.48 \times 10^3$ meters $\approx 3 \times 10^7$ meters $\approx 3 \times 10^4$ kilometers.

PROBLEM 1B

I approach the black hole of mass value $N_{\mathrm{Suns}} = 10^2$. What is the minimum r_{comfort} of the circular orbit in which I will feel comfortable?

SOLUTION 1B

The long horizontal arrow to the right at $N_{\mathrm{Suns}} = 10^2$ in Figure 8 crosses the "comfort curve" at $r_{\mathrm{comfort}}/M \approx 93$, indicated by point B in Figure 8.

PROBLEM 1C

I approach the monster black hole in the center of our galaxy, for which $N_{\mathrm{Suns}} \approx 4 \times 10^6$. Assume (incorrectly) that this monster black hole is not spinning. What is the approximate value of r_{comfort} for this circular orbit?

SOLUTION 1C

The number $M/M_{\mathrm{Sun}} = 4.1 \times 10^6$ is point C on the curve in Figure 8. You will be comfortable in an orbit of approximately $r_{\mathrm{comfort}}/M = 3$

PROBLEM 1D

The robot satellite released by the spaceship at $r_{\mathrm{comfort}}/M = 20$ in Problem 1A is made small and hardened in various ways to withstand tidal accelerations 10^4 times as great as that for which a human will be comfortable. What is the value of r_{comfort} of the circular orbit in which this probe will continue to operate?

SOLUTION 1D

Look at equation (34). The black hole remains the same, so the ratio M/M_{Sun} on the left side remains the same. Therefore the right side must remain the same. When g_{E} in the denominator on the right side *increases* by a factor of 10^4, then its square root contribution to the right side *decreases* by the factor 10^2. To compensate, the square root of the square-bracket expression must *increase* by the factor 10^2. The vertical arrow in the figure extends upward by this factor of 10^2. The leftward horizontal arrow finds r_{conf}/M, for the "comfort orbit" of the robot. This $r_{\mathrm{comfort}}/M \approx 3$ for the robot is at almost the minimum r-value for an unstable circular orbit.

9.7 ■ APPENDIX: KILLER TIDES

₄₅₅

₄₅₆ *Avoid spaghettification!*

₄₅₇ The dangers experienced by Beowulf and other explorers near a neutron star

Size of local
inertial frame
limited by tides.

₄₅₈ should not surprise us. Objects near to one another in curved spacetime can
₄₅₉ experience relative accelerations. Section 1.11 described these "tidal
₄₆₀ accelerations" that limit the size of a local inertial frame. At locations near to
₄₆₁ one another on Earth's surface, these relative accelerations are too small for us
₄₆₂ to notice in everyday life. In contrast, near a neutron star or a black hole
₄₆₃ relative tidal accelerations at different locations on a single human body can
₄₆₄ injure or kill. We call such different accelerations **killer tides**.
₄₆₅   In principle, you can derive the following tidal accelerations using only
₄₆₆ basic tools for the motion of a stone: the metric plus the Principle of Maximal
₄₆₇ Aging. This process, however, is an algebraic nightmare, so we simply quote
₄₆₈ results obtained with the use of a more advanced general-relativistic formalism.

₄₆₉ **TIDES DURING RADIAL MOTION**

Radial motion:
Newton's tidal
accelerations
are valid.

₄₇₀ Surprise! For the special cases of an observer either at rest in global
₄₇₁ coordinates near a black hole or moving radially toward or away from it, local

472 tidal effects predicted by general relativity are identical to those predicted by
473 Newton. Write Newton's expression for gravitational acceleration in the
474 radially outward or local y-direction due to a point or spherically symmetric
475 source. In unitless coordinates:

$$g_y = -\frac{M}{r^2} \qquad \text{(Newton)} \qquad (36)$$

476 Take the differential of this to measure radial tidal effects and write the result
477 in the approximate form for local frame measurements:

$$\Delta g_{\text{local},y} \approx \frac{2M}{\bar{r}^3}\Delta r \approx \frac{2M}{\bar{r}^3}\Delta y_{\text{local}} \qquad \text{(Newton)} \qquad (37)$$

478 The final step, equating Δr to Δy_{local}, makes sense only for Newton; in
479 general relativity the relation between global increment Δr and local frame
480 increment Δy_{local} depends on the position and motion of the local frame in
481 global coordinates. Nevertheless—surprise again!—the full general relativity
482 analysis also yields the last expression in (37). To show this is difficult. The
483 following boxed three equations tell us the tidal accelerations in the three
484 directions in the inertial frame.

$$\Delta g_{\text{local},y} \approx \frac{2M}{\bar{r}^3}\Delta y_{\text{local}} \qquad (38)$$

$$\Delta g_{\text{local},x} \approx -\frac{M}{\bar{r}^3}\Delta x_{\text{local}} \qquad (39)$$

$$\Delta g_{\text{local},z} \approx -\frac{M}{\bar{r}^3}\Delta z_{\text{local}} \qquad (40)$$

Subscript "local" means *any* local frame at rest or moving

radially inward or outward in global rain coordinates.

485

Spaghettification: radial stretch plus tangential compression

486   A radially-diving observer suffers not only stretching in the radial
487 direction, but also compression in tangential directions as her descending body
488 funnels into an ever-narrowing local space. Negative signs in (39) and (40)
489 reflect this compression. We give the light-hearted name **spaghettification** to
490 the physical result of these combined stretch and compression tidal effects:
491 lengthwise extension combined with transverse compression. Sample Problem
492 2 carries out a Newtonian analysis of gravity gradients (tides), whose results
493 turn out to be identical in form to general relativistic results (38) through (40).
494   Expressions (38) through (40) shrink to become calculus expressions (44)
495 at a point. Every approximate equation in this section can lead to a similar
496 calculus expression. We keep the Δ notation, however, to remind us that we
497 deal here with a local frame of finite extent.
498   Now apply equations (38) through (40) to a local *inertial* frame. A liquid
499 drop of nearly incompressible fluid, such as water or mercury, has a surface
500 tension that tends to minimize surface area, which makes the droplet spherical

---

Sample Problem 2. Newton's tidal components

Derive expressions similar to (38) through (40) for Newton's case, in the calculus limit.

SOLUTION:

This is one of only two places in this book where we use vector expressions and partial derivatives. Represent unit vectors in the x, y, and z directions by $\hat{x}$, $\hat{y}$, and $\hat{z}$, respectively. Use this notation to write (36) as a vector equation:

$$\mathbf{g} = -\frac{M\,(x\hat{x} + y\hat{y} + z\hat{z})}{(x^2 + y^2 + z^2)^{3/2}} \qquad \text{(Newton)} \qquad (41)$$

Each component of this vector has the algebraic form:

$$g_q = -\frac{Mq}{(x^2 + y^2 + z^2)^{3/2}} \qquad (42)$$

where q stands for any coordinate x, y, or z. Take the partial derivates similar to the general relativistic equations (38) through (40). You can show that the results also have the same form for all three components:

$$\frac{\partial g_q}{\partial q} = -\frac{M}{r^3} + \frac{3Mq^2}{r^5} \qquad (q \to x, y, z) \qquad (43)$$

We want expressions for these partial derivatives at global coordinate r in flat spacetime. Take y to be along the radial direction, so at that point $y = r$, while $x = z = 0$. Equations (43) become:

$$\frac{\partial g_x}{\partial x} = -\frac{M}{r^3} \qquad \text{(Newton)} \qquad (44)$$

$$\frac{\partial g_y}{\partial y} = -\frac{M}{r^3} + \frac{3M}{r^3} = +\frac{2M}{r^3}$$

$$\frac{\partial g_z}{\partial z} = -\frac{M}{r^3}$$

Inspection shows that equations (44) have the same form as equations (38) through (40).

All radial speeds give same local tidal accelerations.

501 in an inertial frame. Equations (38) through (40) show us that for radial
502 motion, the drop will be distorted into the shape of a throat lozenge or smooth
503 potato—technical term: **prolate spheroid**—shown in Figure 9.
504   Equations (38) through (40) are valid for *all possible* radial
505 speeds—including rest—for example a local inertial frame launched in any of
506 the following ways:

507   • **Local rain frame:** Local inertial frame dropped from rest far away
508   (Box 4, Section 7.4).

509   • **Local hail frame:** Local inertial frame hurled radially inward from far
510   away with any initial local shell speed.

511   • **Local drip frame:** Local inertial frame dropped from rest at any initial
512       $r_0 > 2M$.

513 All of these are radially-moving *local free-fall frames* (Section 2.1). Taken
514 together, free-fall frames result in every possible inward or outward radial

Radial free-fall frames

515 speed of the radially moving frame as measured by a shell observer at any
516 given average $\bar{r}$. General relativity provides results independent of radial speed
517 in (38) through (40), but the tools developed in this book are not sufficient to
518 explain the reason for this result.
519   Notice that equations (38) through (40) satisfy the equation

$$\frac{\Delta g_{\text{local},y}}{\Delta y_{\text{local}}} + \frac{\Delta g_{\text{local},x}}{\Delta x_{\text{local}}} + \frac{\Delta g_{\text{local},z}}{\Delta z_{\text{local}}} \approx 0 \qquad (45)$$
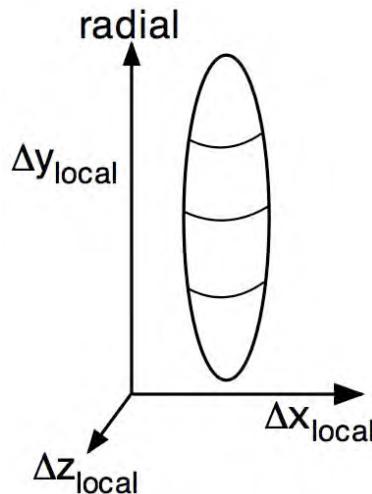
FIGURE 9   Schematic diagram of tide-induced shape for an incompressible liquid drop with surface tension restoring force, observed in a local inertial frame instantaneously at rest or moving radially with respect to a black hole. From the symmetry of the black hole with respect to radial motion, it follows that the tidal squeeze is symmetric perpendicular to the radial direction. Result: the shape is that of an oblong throat lozenge or smooth potato.

Relation among
tidal components

520 This is a general result for tides analyzed by general relativity. In the calculus
521 limit, the approximate equality in (45) becomes mathematically exact, and
522 applies to partial derivatives in (44).

523 **Comment 6. Tides preserve volume.**
524 In the calculus limit, equation (45) expresses a simple and powerful result: The
525 volume of a tiny cloud of free, non-interacting dust particles remains constant as
526 tidal accelerations act on the cloud. This central result is valid even for the far
527 more complicated tidal accelerations near a spinning black hole (Chapter 19).

Tidal effects are
continuous across
event horizon.

528 Notice that equations (38) through (40) are continuous across the event
529 horizon at $r/M = 2$. This result provides additional evidence for our repeated
530 claim that an observer falling through the event horizon experiences a steady
531 increase in tidal effects but no sudden jar or jolt there. Indeed, from evidence
532 internal to her local frame the diver cannot tell when she passes radially
533 inward through the event horizon.

534 **TIDES DURING TANGENTIAL MOTION**

Tangential motion:
tidal accelerations
differ from Newton's.

535 An observer moving in the r, ϕ plane streaks through a local shell frame in the
536 tangential, or Δx_{shell}, direction with shell velocity $v_{\text{shell},x}$. In the following
537 equations, only the factor $M/\bar{r}^3$ reminds us of the corresponding Newtonian
538 analysis in equation (37). *For motion along the tangential* $\pm \Delta x_{\text{shell}}$ *directions:*
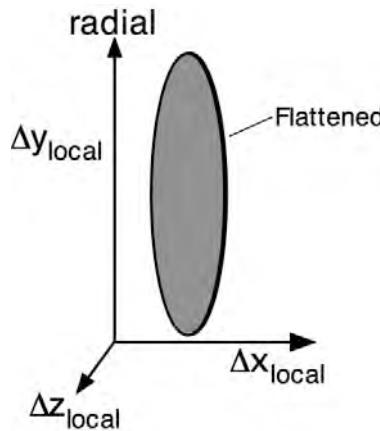
FIGURE 10 Schematic diagram of tide-induced shape for an incompressible liquid drop with surface tension restoring force, observed in a local *inertial* frame that moves in either direction along a Δx_{shell} tangential line. This figure shows results for high tangential speed $v_{\text{shell},x}$: both the tidal stretch in the Δy_{shell} direction and the tidal squeeze in the Δz_{shell} direction are huge, much greater than the tidal squeeze in the Δx_{local} direction. The resulting shape: a thin ribbon with rounded ends lying in the Δx_{shell}, Δy_{shell} plane.

$$\Delta g_{\text{local},y} \approx \left(\frac{1 + v_{\text{shell},x}^2/2}{1 - v_{\text{shell},x}^2} \right) \frac{2M}{\bar{r}^3} \Delta y_{\text{local}} \qquad (46)$$

$$\Delta g_{\text{local},x} \approx -\frac{M}{\bar{r}^3} \Delta x_{\text{local}} \qquad (47)$$

$$\Delta g_{\text{local},z} \approx -\left(\frac{1 + 2v_{\text{shell},x}^2}{1 - v_{\text{shell},x}^2} \right) \frac{M}{\bar{r}^3} \Delta z_{\text{local}} \qquad (48)$$

Subscript "local" means *any* local frame moving tangentially

in either direction in global coordinates.

539

540   Notice that equation (47) is the same as equation (39) for radial motion, while
541   the equations for the other two directions simply multiply the radial results by
Limiting cases   542   coefficients that depend on $v_{\text{shell},x}^2$. In the low-speed limit ($v_{\text{shell},x}^2 \ll 1$), these
for tangential   543   equations also reduce to the radial ones (38) and (40). Finally, note that as
motion   544 $v_{\text{shell},x}$ increases toward the speed of light, the y component leads to radical
545   stretching, while the z component leads to much greater tangential
546   compression than that in the Δx_{local} direction.
547   Expressions (46) through (48) also satisfy the general relation (45) among
548   the local components of gravity gradient, which preserves the volume of a tiny
549   dust cloud moving in the map tangential direction.
550   For a local *inertial frame*, the result is the tidal distortion of a drop of
551   water or liquid mercury into a flat ribbon with rounded ends, shown in Figure

552 10 for tangential motion. Equations (46) through (48) are correct for *any* value
553 of $v_{\text{shell},x}$, not just the value of a stone's local shell speed when it is in a
554 circular orbit. For example, a stone that approaches a black hole from far away
555 and returns to far away will travel tangentially at its point of closest approach;
556 these three equations apply at this point.
557   Section 9.3 applies these results to find the minimum-r circular orbit for
558 human comfort.

559

QUERY 8. Departure from Newton's gravity gradient
Expressions in parentheses on the right sides of (46) and (48) are a measure of the departure of
Einstein's gravity gradients from those predicted by Newton. Temporarily call these expressions
Einstein multipliers.

A. For what value of $v_{\text{shell},x}$ does the largest of the Einstein multipliers become "significant," which
   we define as the value 1.1?

B. For what value of $v_{\text{shell},x}$ does the largest of the Einstein multipliers become "large," which we
   define as the value 10?

C. Exercise 5 in Chapter 1 analyzes the highest energy cosmic ray so far detected, with an energy
   of 3×10^{20} electron volts. Let this cosmic ray be a speeding proton (mass = 1.63×10^{-27}
   kilogram = 9.38×10^8 electron-volts) that streaks tangentially past Earth just above its
   atmosphere, about 100 kilometers above the surface. Estimate the value of the largest Einstein
   multiplier in this case. Hint: Define $v_{\text{shell},x} \equiv 1 - \delta$, then use our approximation formula from
   inside the front cover to redefine the Einstein multipliers in terms of δ.

D. The proton is a quantum particle; its "radius" is not a classical quantity. Nevertheless, estimate
   the tidal stress on the proton cosmic ray of Part C: Assume this proton radius to be 10^{-15}
   meter. What are the tidal accelerations at the surface of the "fastest proton" moving
   tangentially above Earth's atmosphere?

E. Repeat Part D for the "fastest proton" skimming past the surface of a neutron star with
   $r/M = 10$ kilometers.

580

9.8 ■ EXERCISES

1. Smallest circular orbit for a hardened probe around the black hole

583 We harden a probe so that it can withstand K times the maximum
584 comfortable tidal acceleration of a human (Section 9.6). The probe enters a
585 circular orbit around the black hole of mass M in which the tidal acceleration
586 has this maximum. What is the r-value of this circular orbit?

2. The photon (Star Trek) rocket

588 An advanced civilization develops the **photon rocket engine**, one that
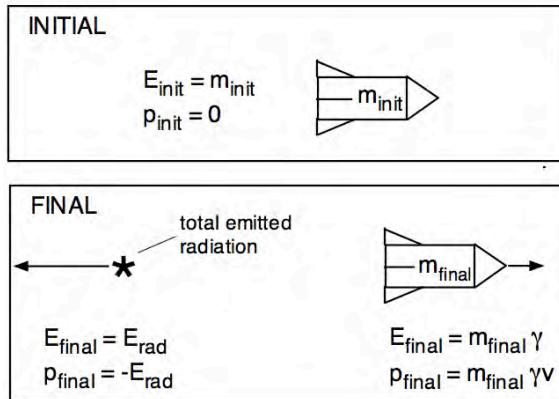589 combines matter and antimatter in a controlled way to yield only photons

FIGURE 11 Exercise 2. Diagram showing initial and final states of a photon rocket that emits only radiation.

(high-energy gamma rays), all of which it directs out the rear of the rocket. The photon rocket engine is the most efficient in the sense that it produces the greatest possible change of velocity for a given fractional change in mass of the rocket ship. Analyze the photon rocket using special relativity, including the definition $\gamma \equiv (1 - v^2)^{-1/2}$.

A. Write down the energy and momentum conservation laws using Figure 11.
B. Combine the conservations laws, show that $\gamma v = (\gamma^2 - 1)^{1/2}$, and derive the equation for the *mass ratio*:

$$\frac{m_{\text{init}}}{m_{\text{final}}} = \gamma + (\gamma^2 - 1)^{1/2} \quad \text{(photon rocket, flat spacetime)} \quad (49)$$

where m_{init} is the initial mass of the rocket ship.
C. Find the mass ratio for $\gamma = 10$
D. Show that the result of Part C is an example of the approximation

$$\frac{m_{\text{init}}}{m_{\text{final}}} \approx 2\gamma \quad \text{(when } \gamma^2 \gg 1\text{)} \quad \text{(photon rocket, flat spacetime) (50)}$$

3. Newton's Tangential Tidal Displacement Near Earth.

Brave Monica Sefner "walks the plank" at the top of the 828-meter-tall Dubai Tower, Burj Khalifa (Figure 12), on which she moves horizontally outward to a point that clears the base of the tower. Then she steps off the plank attached to a bungee cord and falls freely for 600 meters, at which point the cord "takes hold" and slows her to a stop before she reaches the ground. As she leaves the plank, Monica stretches out her arms and releases from rest two marbles

FIGURE 12   Exercise 3. DubaiTower, 828 meters high.
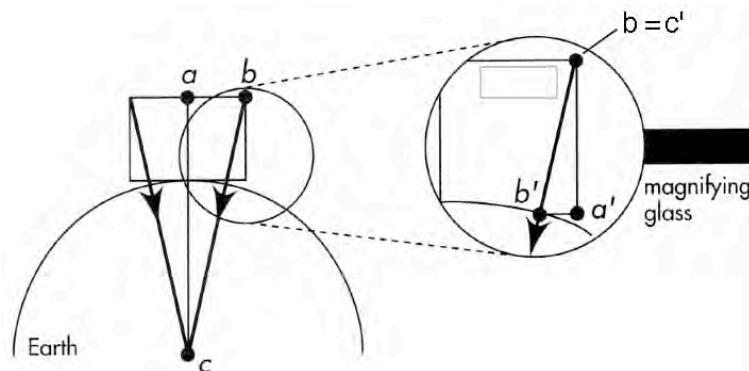
FIGURE 13   Exercise 3. Construction to analyze tangential tidal acceleration of radially falling marbles in Newton's mechanics. Not to scale, and with gross differences in relative scale of different parts of the diagram.

609 initially 2 meters apart horizontally. Just before the end of her 600-meter free
610 fall, how much will the measured separation between these marbles have
611 decreased? Will Monica be able to measure this decrease in separation? To
612 answer these questions, use the following method of similar triangles (Figure
613 13) or your own method.

614   Assume that the air neither slows down nor deflects either marble from
615   its straight-line course. Then each marble falls from rest toward the
616   center of Earth, as indicated by arrows in Figure 13. Solve the problem
617   using the ratio of sides of similar triangles abc and $a'b'c'$. These triangles

618   are upside down with respect to one another, but they are similar
619   because their respective sides are parallel. We know the lengths of some
620   of these sides (some greatly exaggerated in the figure): Side $b'c' = 600$
621   meters; side bc is effectively equal to the r-coordinate of Earth; side
622    $ab = 1$ meters equals half of the original separation of the marbles; side
623    $a'b'$ equals *half the change* in their separation after a drop of 600 meters.

A. Use the ratio of sides of similar triangles to find the "half change" in
   separation as the two marbles fall 600 meters. From this result, find the
   entire change in separation between the marbles.

B. Suppose that, as she steps off the plank, Monica releases the two
   marbles from rest with a *vertical* separation of 2 meters. From
   Newton's equations (36) and (37), find the increase in separation of two
   marbles after they fall 600 meters, under the assumption that the
   marbles fall in a vacuum.)

C. Re-derive your result of Part A using the simpler Part B plus equation
   (45).

4. Measure your global radial coordinate r near a black hole?

635   You are the captain of a spaceship with rockets blasting as you descend slowly
636   toward a black hole along a radial line. In effect, you stand for a minute on
637   each shell, then step downward sequentially to the next shell below. From
638   earlier observations you know the value of the black hole mass M and would
639   like to measure your map r-coordinate in order to be sure you are not near the
640   event horizon.

A. Describe how you can determine r from the initial acceleration of a test
   particle as you descend.

B. Oops! Is there a paradox here? You have measured a map quantity, r,
   using observations on a local shell. Isn't that illegal?

5. Spaceship approach at relativistic speed

646   The present chapter assumes that the approaching spaceship moves
647   slowly—not at relativistic speed—with respect to the black hole, so that
648    $E/m \approx 1$. But the captain of the approaching spaceship does not want to
649   waste valuable rocket fuel to slow down in order to apply the analysis of this
650   chapter. She decides not to reduce the large value of her map energy E/m
651   (with respect to the black hole) and instead to use her main thrusters to
652   adjust the value of her map angular momentum $L/(mM)$ so that she moves
653   directly to a knife-edge orbit. If the rocket thrust that increase L/m also
654   increases E/m, no problem: Just use the final value of E/m in what follows.

655   A. For a large value of map energy $E/m \gg 1$, the r-value of the knife-edge
656   orbit is only slightly greater than 3M. Set $r/M = 3(1 + \delta)$ in (8). Show
657   that:

$$\frac{E}{m} \approx \frac{1}{3\delta^{1/2}} \qquad\qquad (E/m \gg 1, \text{ knife-edge orbit}) \qquad (51)$$

658   so that for the given large value of E/m,

$$\delta^{1/2} \approx \frac{m}{3E} \qquad\qquad (E/m \gg 1, \text{ knife-edge orbit}) \qquad (52)$$

659   B. Show that for this case, equation (6) for the knife-edge orbit becomes:

$$\frac{L}{mM} \approx \left(\frac{3}{\delta}\right)^{1/2} = 3^{3/2}\frac{E}{m} \qquad (E/m \gg 1, \text{ knife-edge orbit}) \quad (53)$$

660   C. When observations are complete, how does the commander move away
661   from the black hole? Give a general description of this maneuver; don't
662   sweat the details.

663 **6. Swoop Orbit**

664 Figure 14 shows the effective potential for a so-called **swoop orbit** of a stone
665 whose map energy E/m is slightly smaller than that of the effective potential
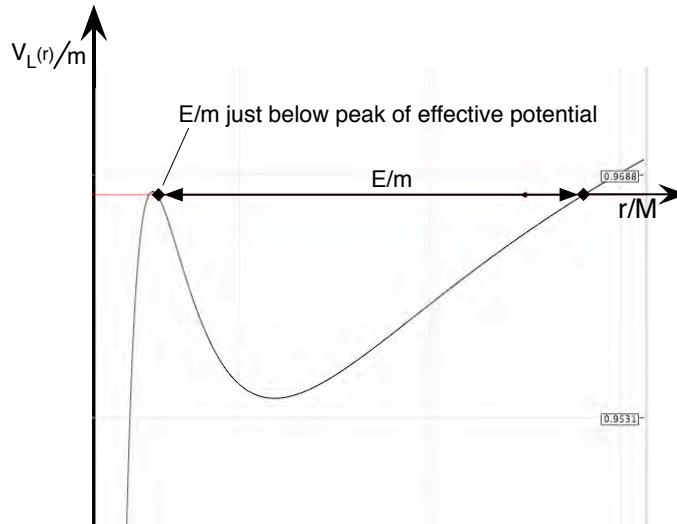666 peak at small r-value.

FIGURE 14   Exercise 6: Effective potential for the **swoop orbit** of a stone with map energy E/m just below the (left-hand peak) of the effective potential.

A. Make a rough sketch of the swoop orbit on the $[r, \phi]$ slice. *Optional:* Use interactive softward GRorbits to create and print this swoop orbit.

Luc Longtin is a junior engineer at the Space Agency. He claims that with a small rocket thrust he can put the entire incoming spaceship into a swoop orbit that oscillates between $r = 4M$ and $r = 100M$. This will allow direct observations from the spaceship at r-values between these two limits, completely eliminating the need for probes.

The Space Agency rejects Luc's plan as too risky. Luc invites you, the Chief Engineer, to a bar where he tries to convince you to that the Space Agency should reverse its decision and use his plan. Luc lays out his proposal as follows:

B. Luc begins, "Look at the effective potential for $L/(mM) = 4$ in Figure 6. The inner peak of this effective potential is at $r = 4M$ with $E/m = 1$ and the spaceship approaches from far away with $E/m = 1 + \epsilon$, where $\epsilon = 0.001$. My plan is that when the spaceship reaches, say $r = 20$, it uses a tiny rocket thrust to flip its map energy to $E/m = 1 - \epsilon$ without changing its angular momentum (so the effective potential does not change). Let engineers worry about details of that thrust; just look at the result. The spaceship enters a swoop orbit that bounces off the effective potential peak just outside $r = 4M$. At that bounce, $dr/d\tau = 0$, so equation (17) in Section 8.4 becomes"

$$\frac{dr}{d\tau} = 0 = \left(\frac{E}{m}\right)^2 - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{L^2}{m^2 r^2}\right) \tag{54}$$

$$0 = (1 - \epsilon)^2 - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{16M^2}{r^2}\right) \tag{55}$$

$$0 = 32\left(\frac{M}{r}\right)^3 - 16\left(\frac{M}{r}\right)^2 + 2\left(\frac{M}{r}\right) - [1 - (1 - \epsilon)^2] \tag{56}$$

Fill in the steps between (55) and (56).

C. Luc continues, "We set up equation (56) for the bounce point near $r = 4M$. But this equation has only global map quantities in it, so is also correct for the bounce point at the large r-value at the outward end of the swoop orbit. At this large r-value, the first term on the right of (56) is small compared to the other terms, so neglect this first term. What remains is a quadratic in the small quantity M/r. Solve this quadratic to show that the only acceptable solution for large r/M is $M/r = \epsilon$ or $r = M/\epsilon = 100M$ for the right-hand bounce point of the swoop orbit."

Verify Luc's calculations.

C. Luc concludes, "So a very small rocket thrust installs the entire incoming spaceship in a swoop orbit that moves in and out between

701    $r = 100M$ and an r-value slightly greater than $r = 4M$. No need for
702   those silly probes. Astronauts can make observations in this orbit as
703   long as they want as they move in and out. When they finish, a small
704   rocket thrust similar to that described in Item B (during the outgoing
705   portion of its orbit) flips the spaceship map energy back to
706    $E/m = 1 + \epsilon$, so the spaceship escapes the black hole."

707   Do you agree with this part of Luc's plan?

708   Will you recommend Luc's program to the Space Agency?

9.9■ REFERENCES

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726   Most of the orbit plots in this chapter were made using the interactive
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728   Download File Name: Ch09Orbiting160404v1.pdf