Chapter 7. Inside the Black Hole

- *Why would anyone volunteer to dive to the center of a black hole?*
- *Why does everything inside the event horizon inevitably move to smaller r?*
- *How massive must a black hole be so that 20 years pass on my wristwatch between crossing the event horizon and arrival at the crunch point?*
- *How can I construct a local inertial frame that is valid inside the event horizon?*
- *What do I see ahead of me and behind me as I approach the crunch point?*
- *Is my death quick and painless?*

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CHAPTER 7

Inside the Black Hole

Edmund Bertschinger & Edwin F. Taylor *

Alice had not a moment to think about stopping herself before she found herself falling down what seemed to be a very deep well. Either the well was very deep, or she fell very slowly, for she had plenty of time as she went down to look about her, and to wonder what was going to happen next. First she tried to look down and make out what she was coming to, but it was too dark to see anything . . . So many out-of-the-way things had happened lately that Alice had begun to think that very few things indeed were really impossible.

—Lewis Carroll, *Alice in Wonderland*

7.1 ■ INTERVIEW OF A DIVING CANDIDATE

Few things are really impossible.

So you are applying to be a member of the black hole diving research group.

Yes.

Have you personally had experience diving into black holes?

This question is a joke, right?

Why do you want to be part of this diving group, since your research results cannot be reported back to us outside the event horizon?

We want to see for ourselves whether or not our carefully-studied predictions are correct. You know very well that 27 percent of qualified Galaxy Fleet personnel volunteered for this mission.

Tell me, why doesn't the black-hole diving group use local shell coordinates to make measurements inside the event horizon?

*Draft of Second Edition of *Exploring Black Holes: Introduction to General Relativity*

> *Inside the event horizon no one can build a spherical shell that stays at constant r in Schwarzschild coordinates. Instead we make measurements in a series of local inertial frames that can exist anywhere except on the singularity.*

Then how will you measure your r-coordinate without a spherical shell?

> *We track the decreasing value of r by measuring the decreasing separation between us and a test particle beside us that is also diving radially inward.*

What clocks will you use in your experiments?

> *Our wristwatches.*

When does your diving group cross the event horizon?

> *As measured on whose clock?*

You *are* savvy. When does your diving group cross the event horizon as read on your wristwatches?

> *Zeroing different clocks in different locations is arbitrary. The Astronautics Commission has a fancy scheme for coordinating the various clock readings, mostly for convenience in scheduling. Want more details?*

Not now. Is there any service that we on the outside can provide for your diving group once you are inside the event horizon?

> *Sure. We will welcome radio and video bulletins of the latest news plus reports of scientific developments outside the event horizon.*

And will your outgoing radio transmissions from inside the event horizon change frequency during their upward transit to us?

> *Another joke, I see.*

Yes. Does your personal—ah—end seem mercifully quick to you?

> *The terminal "spaghettification" will take place in a fraction of a second as recorded on my wristwatch. Many of you outside the event horizon would welcome assurance of such a quick end.*

What will you personally do for relaxation during the trip?

> *I am a zero-g football champion and grandmaster chess player. Also, my fiancé has already been selected as part of the team. We will be married before launch.*

[We suppress the transcript of further discussion about the ethical and moral status of bringing children into the diving world.]

Box 1. Eggbeater Spacetime?

Being "spaghettified" as you approach the center of a black hole is bad enough. But according to some calculations, your atoms will be scrambled by violent, chaotic tidal forces before you reach the center—especially if you fall into a young black hole.

The first theory of the creation of a black hole by J. Robert Oppenheimer and Hartland Snyder (1939) assumed that the collapsing structure is spherically symmetric. Their result is a black hole that settles quickly into a placid final state. A diver who approaches the singularity at the center of the Oppenheimer-Snyder black hole is stretched with steadily increasing force along the r-direction and compressed steadily and increasingly from all sides perpendicular to the r-direction.

In Nature an astronomical collapse is rarely spherically symmetric. Theory shows that when a black hole forms, the asymmetries exterior to the event horizon are quickly radiated away in the form of gravitational waves—in a few seconds measured on a distant clock! Gravitational radiation captured inside the event horizon, however, evolves and influences spacetime inside the black hole.

So what happens? There is no way to verify any predictions about events inside the event horizon (Objection 1), but that does not stop us from making them! Vladimir Belinsky, Isaac Markovich Khalatnikov, Evgeny Mikhailovich Lifshitz, and independently Charles Misner discovered that Einstein's equations predict more than one kind of singularity.

Their theory says that as a diving observer approaches the center point, spacetime can oscillate chaotically, squeezing and stretching the poor traveler in random directions like an electric mixer (eggbeater). These oscillations increase in both amplitude and frequency as the astronaut approaches the singularity of the black hole. Any physical object, no matter what stresses it can endure, is necessarily utterly destroyed at an eggbeater singularity.

However, there is some theoretical evidence that eggbeater oscillations will die away, so an astronaut who waits a while to dive after the black hole has formed may not encounter them. Before these eggbeater oscillations die away—if they do—spacetime in the chaotic regions is definitely NOT described by the Schwarzschild metric!

In the present chapter we assume the non-spinning black hole under exploration is an ancient one and that we can ignore possible eggbeater oscillations of spacetime. We predict (and hope!) that as our astronaut colony approaches the center, the "spacetime weather" is clear and calm.

?

Objection 1. *What kind of science are these people talking about? Obviously nothing more than science fiction! No one who crosses the event horizon of a black hole can report observations to the scientific community outside the event horizon. Therefore all observations carried out inside the event horizon—and conclusions drawn from them—remain private communications. Private communication is not science!*

!

Yours is one sensible view of science, but if the "spacetime weather is clear and calm" inside the event horizon (Box 1), then the diving research group may have decades of life ahead of them, as recorded on their wristwatches. They can receive news and science updates from friends outside, view the ever-changing pattern of stars in the heavens (Chapter 12), carry out investigations, discuss observations among themselves, and publish their own exciting research journal.

We recognize that the event horizon separates two communities of investigators with a one-way surface or "membrane." Outsiders cannot receive reports of experiments that test their predictions about life inside the event horizon. They must leave it to insiders to verify or disprove these predictions with all the rigor of a lively in-falling research community. Later chapters on the *spinning* black hole raise the possibility that an explorer might navigate in such a way as to reemerge from the event horizon, possibly into a different spacetime region.

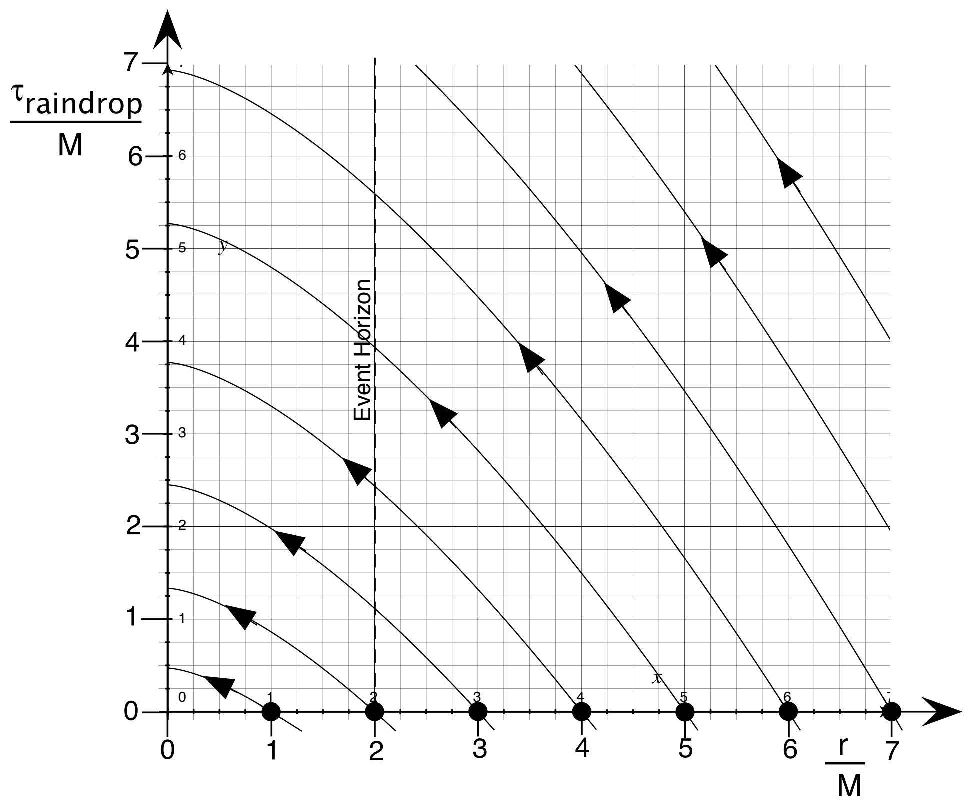

FIGURE 1 Raindrop wristwatch time vs. global r-coordinate from (2) for a series of raindrops that pass $r = M, 2M, ..., 9M$ at $\tau_{\rm raindrop} = 0$ (little filled circles along the horizontal axis). The shapes of these curves are identical, just displaced vertically with respect to one another. Every raindrop moves smoothly across the event horizon when clocked on its wristwatch, but not when tracked with global Schwarzschild t-coordinate (Figure 2).

John A. Wheeler: radical conservatism

Comment 1. Wheeler's "radical conservatism"
John Archibald Wheeler (1911-2008), who co-authored the first edition of *Exploring Black Holes*, rescued general relativity from obscurity in the 1950s and helped to jump-start the present golden age of gravitational physics. He was immensely inventive in research and teaching; for example he adopted and publicized the name *black hole* (initial quote, Chapter 3). Wheeler's professional philosophy was **radical conservatism**, which we express as: *Follow well-established physical principles while pushing each to its extreme limits. Then develop a new intuition!* The black hole—both outside and inside its event horizon—is a perfect structure on which to apply Wheeler's radical conservatism, as we do throughout this book.

Comment 2. Non-spinning vs. spinning black hole
Chapters 2 through 13 describe spacetime around a black hole: a black hole that does not rotate. We call this a **non-spinning black hole**. The Universe is full of black holes that spin; many of them spin very fast, with deep

consequences for their structure and for spacetime around them. We call each of these a **spinning black hole**, the subject of Chapters 17 through 21.

7.2 ■ RAINDROP WORLDLINE

"Raindrops keep fallin' on my head . . ." song by Hal David and Burt Bacharach

We start with the raindrop of Chapter 6. The *raindrop* is a stone (wearing a wristwatch) that drops from initial rest very far from the black hole (Definition 2, Section 6.4). From equation (23) of that section:

$$d\tau_{\text{raindrop}} = -\left(\frac{r}{2M}\right)^{1/2} dr \tag{1}$$

In the following Queries you integrate (1) and apply the result to a raindrop that first falls past an Above r-coordinates r_{A} then falls past a sequence of lower r-coordinates (Figure 1).

QUERY 1. Raindrop wristwatch time lapse between the above r_{A} and lower r

A. Integrate (1) to determine the elapsed raindrop wristwatch time from the instant the raindrop falls past the Above coordinate r_{A} until it passes a sequence of smaller r-coordinates. Express this elapsed raindrop wristwatch time with the notation $[r_{\text{A}} \to r]$ for r-limits.

$$\tau_{\text{raindrop}}\left[r_{\text{A}} \to r\right] = \frac{4M}{3}\left[\left(\frac{r_{\text{A}}}{2M}\right)^{3/2} - \left(\frac{r}{2M}\right)^{3/2}\right] \tag{2}$$

Figure 1 plots this equation for a series of raindrops after each passes through a different given r_{A} at $\tau_{\text{raindrop}} = 0$.

B. What happens to the value of the raindrop wristwatch time lapse in (2) when the initial r_{A} becomes very large? Explain why you are not disturbed by this result.

QUERY 2. Raindrop wristwatch time lapse from event horizon to crunch.
Suppose you ride the raindrop, and assume (incorrectly) that you survive to reach the center. This Query examines how long (on your wristwatch) it takes you to drop from the event horizon to the singularity—the crunch point.

A. Adapt your result from Query 1 to show that:

$$\tau_{\text{raindrop}}[2M \to 0] = \frac{4M}{3} \qquad \text{(event horizon to crunch, in meters)} \tag{3}$$

B. *Predict:* Does every curve in Figure 1 satisfy (3)?

C. Use constants inside the front cover to find the event horizon-to-crunch raindrop wristwatch time *in seconds* for a black hole of mass M/M_{Sun} times the mass of our Sun:

$$\tau_{\text{raindrop}}[2M \rightarrow 0] = 6.57 \times 10^{-6} \frac{M}{M_{\text{Sun}}} \qquad \text{(event horizon to crunch, in seconds)} \tag{4}$$

D The monster black hole at the center of our galaxy has mass $M \approx 4 \times 10^6 M_{\text{Sun}}$: its mass is about four million times the mass of our Sun. Assume (incorrectly) that this black hole does not spin. How long, in seconds on your wristwatch, will it take you—riding on the raindrop—to fall from the event horizon of this monster black hole to its singularity?

E. **Discussion question:** How can the r-value of the event horizon $r = 2M$ possibly be greater than the wristwatch time $4M/3$ that it takes the raindrop to fall from the event horizon to the singularity? Does the raindrop move faster than light inside the event horizon? (*Hint:* Do global coordinate separations dependably predict results of our measurements?)

QUERY 3. Mass of the "20-year black hole."

The black hole we feature in this chapter has a mass such that it takes 20 years—recorded on the wristwatch of the raindrop—to fall from event horizon to singularity.

A. Find the mass of the "20-year black hole" (a) in meters, (b) as a multiple of the mass of our Sun, and (c) in light-years.

B. An average galaxy holds something like 10^{11} stars of mass approximately equal to that of our Sun. The "20-year black hole" has the mass of approximately how many average galaxies?

C. What is the value of the r-coordinate at the event horizon of the "20-year black hole" in light-years?

Now turn attention to the motion of the raindrop in global Schwarzschild coordinates. Equation (22) in Section 6.4 gives the Schwarzschild map velocity of the raindrop:

$$\begin{aligned}\frac{dr}{dt} &= -\left(1 - \frac{2M}{r}\right)\left(\frac{2M}{r}\right)^{1/2} \qquad \text{(raindrop map velocity)} \\ &= \left(\frac{r}{2M}\right)^{-3/2}\left(1 - \frac{r}{2M}\right)\end{aligned} \tag{5}$$

We want to find $r(t)$, the r-coordinate of the raindrop as a function of the t-coordinate. This function defines a worldline (Section 3.10).

To find the worldline of the raindrop, manipulate (5) to read:

$$dt = \frac{\left(\dfrac{r}{2M}\right)^{3/2} dr}{1 - \dfrac{r}{2M}} = \frac{4Mu^4 du}{1 - u^2} \qquad \text{(raindrop)} \tag{6}$$

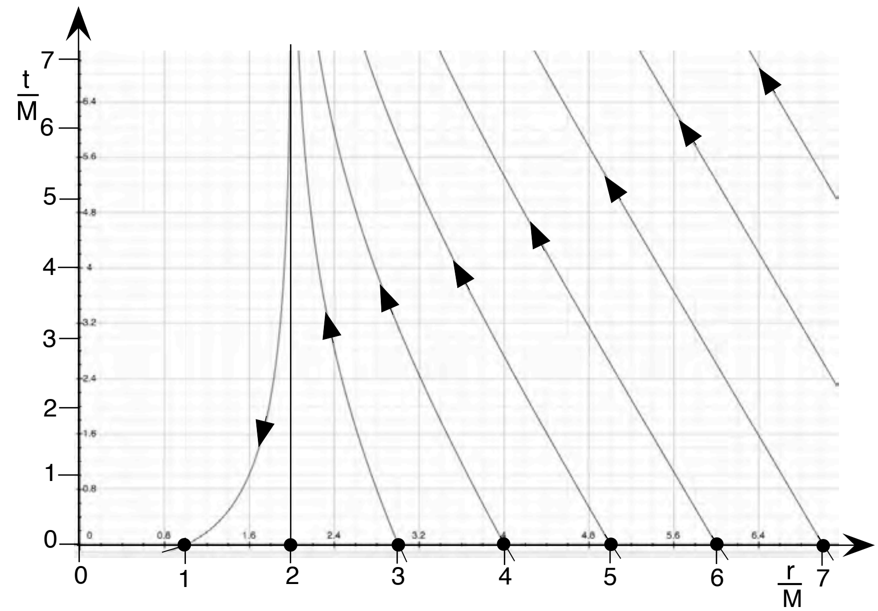

FIGURE 2 Schwarzschild worldlines of raindrops from (10), plotted on the $[r,t]$ slice. These particular raindrops pass $t_{\rm A}/M = 0$ at different values of $r_{\rm A}/M$ (filled dots along the horizontal axis). The curves for $r_{\rm A}/M > 2$ are identical in shape, simply displaced vertically with respect to one another. These worldlines are not continuous across the event horizon (compare Figure 1).

where the expression on the right side of (6) results from substitutions:

$$u \equiv \left(\frac{r}{2M}\right)^{1/2} \quad \text{so} \quad du = \frac{1}{4M}\left(\frac{r}{2M}\right)^{-1/2} dr \quad \text{and} \quad dr = 4Mudu \tag{7}$$

From a table of integrals:

$$\int \frac{u^4 du}{1-u^2} = -\frac{u^3}{3} - u + \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right| \tag{8}$$

Integrate (6) from $u_{\rm A}$ to u, where A stands for Above. The integral of (6) becomes:

$$t - t_{\rm A} = 4M\left[\frac{u_{\rm A}^3}{3} - \frac{u^3}{3} + u_{\rm A} - u + \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right| - \frac{1}{2}\ln\left|\frac{1+u_{\rm A}}{1-u_{\rm A}}\right|\right] \quad \text{(raindrop)} \tag{9}$$

Substitute the expression for u from (7) into (9):

$$t - t_{\rm A} = \frac{4M}{3}\left[\left(\frac{r_{\rm A}}{2M}\right)^{3/2} - \left(\frac{r}{2M}\right)^{3/2} + 3\left(\frac{r_{\rm A}}{2M}\right)^{1/2} - 3\left(\frac{r}{2M}\right)^{1/2}\right. \tag{10}$$

$$\left. + \frac{3}{2}\ln\left|\frac{1+\left(\frac{r}{2M}\right)^{1/2}}{1-\left(\frac{r}{2M}\right)^{1/2}}\right| - \frac{3}{2}\ln\left|\frac{1+\left(\frac{r_{\rm A}}{2M}\right)^{1/2}}{1-\left(\frac{r_{\rm A}}{2M}\right)^{1/2}}\right|\right] \quad \text{(raindrop)}$$

Equation (10) is messy, but the computer doesn't care and plots the curves in Figure 2 for $t_{\mathrm{A}} = 0$ and $r_{\mathrm{A}}/M = 1$ through 9.

Objection 2. *Wait! The t/M versus r/M worldline in Figure 2 tells us that the raindrop takes an unlimited t to reach the event horizon. Do you mean to tell me that our raindrop does not cross the event horizon?*

Recall our "strong advice" in Section 5.6: "To be safe, it is best to assume that global coordinate separations do not have any measured meaning." The worldlines in Figure 2 that rise without limit in t-coordinate as $r/M \to 2^+$ (from above) do not tell us directly what any observer measures. In contrast, the observer riding on the raindrop reads and records her wristwatch time τ as she passes each shell. At each such instant on her wristwatch, she also records her direct reading of the r-coordinate stamped on the shell she is passing. When timed on her wristwatch, the raindrop passes smoothly inward across the event horizon (Figure 1).

Objection 3. *There is still a terrible problem with Figure 2. Why does the worldline of the raindrop that somehow makes it inward across $r/M = 2$ run* ***backward*** *in the Schwarzschild t-coordinate?*

We saw this earlier in the light-cone diagram of Figure 8 in Section 3.7, in which the t-coordinate can run backward along a worldline. However, that has no measurable consequence. You cannot grow younger by falling into a black hole. Sorry! The global t-coordinate is *not* time. Does the idea of backward-running global t-coordinate along a worldline make you uncomfortable? Get used to it! In contrast, the time you read on your wristwatch *always* runs forward along your worldline, in particular along the raindrop worldline (Figure 1). In Section 7.4 we develop a set of **global rain coordinates** that not only labels each event—which we require of every set of global coordinates—but also yields predictions more comfortable to our intuition about the "respectable sequence" of global coordinates along a worldline. This (arbitrary) choice of global coordinates is purely for our own convenience: Nature doesn't care!

7.3 ■ THE LOCAL RAIN FRAME IN SCHWARZSCHILD COORDINATES

Carry out experiments as you pass smoothly through the event horizon

Does passing through the event horizon disturb local experiments that we may be conducting during this passage? To answer this questions go step by step from the raindrop to the local inertial rain frame and (in Section 7.4) from the local frame to a new global description. Start with local shell coordinates outside the horizon and use special relativity to derive local rain frame coordinates.

Equations (9) through (11) in Section 5.7 give us local shell coordinates expressed in Schwarzschild global coordinates:

$$\Delta t_{\text{shell}} \equiv \left(1 - \frac{2M}{\bar{r}}\right)^{1/2} \Delta t \qquad \text{(from Schwarzschild)} \tag{11}$$

$$\Delta y_{\text{shell}} \equiv \left(1 - \frac{2M}{\bar{r}}\right)^{-1/2} \Delta r \tag{12}$$

$$\Delta x_{\text{shell}} \equiv \bar{r}\Delta\phi \tag{13}$$

Now use the Lorentz transformation equations of Section 1.10 to find the local time lapse Δt_{rain} in an inertial frame in which the raindrop is at rest. With respect to the local shell frame, the raindrop moves with velocity v_{rel} in the $-\Delta y_{\text{shell}}$ direction. From equation (19) in Section 6.4:

$$v_{\text{rel}} = -\left(\frac{2M}{r}\right)^{1/2} \quad \text{so} \quad \gamma_{\text{rel}} \equiv \frac{1}{(1 - v_{\text{rel}}^2)^{1/2}} = \left(1 - \frac{2M}{r}\right)^{-1/2} \tag{14}$$

Then from the first of equations (41) in Section 1.10:

$$\begin{aligned} \Delta t_{\text{rain}} &= \gamma_{\text{rel}} \left(\Delta t_{\text{shell}} - v_{\text{rel}} \Delta y_{\text{shell}}\right) \qquad (15) \\ &= \left(1 - \frac{2M}{\bar{r}}\right)^{-1/2} \left[\left(1 - \frac{2M}{\bar{r}}\right)^{1/2} \Delta t + \left(\frac{2M}{\bar{r}}\right)^{1/2} \left(1 - \frac{2M}{\bar{r}}\right)^{-1/2} \Delta r\right] \end{aligned}$$

so that

$$\Delta t_{\text{rain}} = \Delta t + \left(\frac{2M}{\bar{r}}\right)^{1/2} \left(1 - \frac{2M}{\bar{r}}\right)^{-1} \Delta r \tag{16}$$

And from the second of equations (41) in Section 1.10:

$$\begin{aligned} \Delta y_{\text{rain}} &= \gamma_{\text{rel}} \left(\Delta y_{\text{shell}} - v_{\text{rel}} \Delta t_{\text{shell}}\right) \qquad (17) \\ &= \left(1 - \frac{2M}{\bar{r}}\right)^{-1/2} \left[\left(1 - \frac{2M}{\bar{r}}\right)^{-1/2} \Delta r + \left(\frac{2M}{\bar{r}}\right)^{1/2} \left(1 - \frac{2M}{\bar{r}}\right)^{1/2} \Delta t\right] \end{aligned}$$

so that

$$\Delta y_{\text{rain}} = \left(1 - \frac{2M}{\bar{r}}\right)^{-1} \Delta r + \left(\frac{2M}{\bar{r}}\right)^{1/2} \Delta t \tag{18}$$

Finally, from the third of equations (41) in Section 1.10, the shell and rain coordinates transverse to the direction of relative motion have equal values:

$$\Delta x_{\text{rain}} = \Delta x_{\text{shell}} = \bar{r}\Delta\phi \tag{19}$$

The right sides of local rain frame equations (16) and (18) suffer from the same disease as their parent global Schwarzschild coordinates: They blow up at the event horizon. Nevertheless, we can use these *local* coordinate equations to derive a new set of *global* rain coordinates that, at long last, cures this disease and allows us to predict observations made in the local rain frame as we ride smoothly inward across the event horizon and all the way to the singularity.

7.4 ■ GLOBAL RAIN COORDINATES

Convert from Schwarzschild-t to global rain T.

Schwarzschild coordinates are completely legal, but they make us uncomfortable because they do not describe the motion of a stone or light flash inward across the event horizon in a finite lapse of the t-coordinate. So in this section we find a new global coordinate—that we label the T-coordinate—which advances smoothly along the global worldline of a descending stone, even when the stone crosses the event horizon.

Global rain coordinates or "Painlevé-Gullstrand coordinates."

The result is a new set of global coordinates with the old global coordinates r and ϕ but a new T-coordinate. We call this new set of coordinates **global rain coordinates**. They are often called **Painlevé-Gullstrand coordinates** after Paul Painlevé and Alvar Gullstrand who independently developed them in 1921 and 1922, respectively. The present section develops global rain (Painlevé-Gullstrand) coordinates. Section 7.5 uses these new global coordinates to derive the **global rain metric**

?

Objection 4. *Hold on! How can we have two different global coordinate systems for the same spacetime?*

!

For the same reason that a flat Euclidean plane can be described by either Cartesian coordinates or polar coordinates. More than one global coordinate system can describe the same spacetime. Indeed, an unlimited number of global coordinate systems exist for any configuration of mass-energy-pressure (Box 3).

?

Objection 5. *I am awash in arbitrary global coordinates here. What makes practical sense of all this formalism? What can I hang onto and depend upon?*

!

Primary goal: Predict result of local measurements

The answer comes from invariant wristwatch time and invariant ruler distance. These direct observables are outputs of the global metric. In John Wheeler's words, *"No phenomenon is a real phenomenon until it is an observed phenomenon."* Spacetime is effectively flat on a local patch; on that flat patch we use the approximate global metric to derive local coordinates that we choose to be inertial (Section 5.7). The observer makes a measurement and expresses the result in those local

> coordinates. **Every set of global coordinates must lead to the same predicted result of a given measurement.** That is what makes sense of the formalism; that is what you can hang onto and depend on.

The troublemaker in Schwarzschild coordinates is the t-coordinate, which Figure 2 shows to be diseased at the event horizon. The cure is a new global rain coordinate which we call capital T. The other two global rain coordinates, r and ϕ remain the same as the corresponding Schwarzschild coordinates.

> **Comment 3. Why conversion from Schwarzschild to global rain?**
> Most often in this book we simply display a global metric with its global coordinate system without derivation. In what follows, we make a “conversion” of Schwarzschild coordinates to global rain coordinates that leads to the global rain metric. Why this conversion? Why don’t we simply display the global rain metric and its coordinate system? We do this to show that there are two ways to derive a valid global metric. The first way is to submit the (almost) arbitrary global coordinates to Einstein’s equations, which return the correct global metric. The second way is simply to transform the already-validated global metric directly. That conversion does not require Einstein’s equations.

To find the new global T coordinate we do something apparently illegal: We derive it from *local* rain coordinate Δt_{rain} in equation (16). From the beginning of this book we have emphatically declared that you cannot derive global coordinates from local coordinates. Why not? Because in considering any flat local inertial frame, we lose details of the global curvature of the spacetime region. The local Δt_{rainA} from equation (16) is unique to Frame A which depends on the average value $\bar{r}_{\text{A}}$; adjacent Frame B has a different Δt_{rainB} which depends on a different average value $\bar{r}_{\text{B}}$. This leads to a discontinuity at the boundary between these two local frames. The local frame with its local coordinates is useful for us because it allows us to apply special relativity to an experiment or observation made in a limited spacetime region in globally curved spacetime. But the local frame does have this major drawback: We cannot connect adjacent flat frames smoothly to one another in curved spacetime. That keeps us from generalizing from local frame coordinates to global coordinates.

You cannot derive global coordinates from local coordinates. . . .

But there is an exception. To understand this exception, pause for a quick tutorial in the mathematical theory of calculus (invented in the late 1600s by both Isaac Newton and Gottfried Wilhelm Leibniz). In calculus we use differentials, for example dt and dr. The technical name for the kind of differential we use in this book is **exact differential**, sometimes called a **perfect differential**. Formally, an exact differential (as contrasted with an inexact differential or a partial differential) has the form dQ or dT, where Q and T are **differentiable functions**. What is a differentiable function? It is simply a function whose derivative exists at every point in its domain.

. . . except when local coordinates lead to an exact differential.

Question: Are global coordinates differentiable? *Answer:* We choose global coordinates ourselves, then submit them to Einstein’s equations which are differential equations that return to us the global metric. So if one of our chosen global coordinates is not differentiable, it is our own fault. *Conclusion:*

Global coordinates are differentiable.

For a global coordinate to be useful in general relativity, it must be differentiable and thus have an exact differential (except at a physical singularity).

So we purposely choose a set of global coordinates that are differentiable. If we then make a transformation between sets of global coordinates, the transformed global coordinate is also a differentiable function and therefore has an exact differential.

A close look at equation (16) shows that we can turn it into an exact differential, as follows:

$$\lim_{\Delta t \to 0} \Delta t_{\text{rain}} = dt + \frac{df(r)}{dr} dr = d[t + f(r)] \equiv dT \tag{20}$$

where, from (16), $T = t + f(r)$ is a differentiable function with

$$\frac{df(r)}{dr} = \left(\frac{2M}{r}\right)^{1/2} \left(1 - \frac{2M}{r}\right)^{-1} \tag{21}$$

Equation (20) is immensely significant. It tells us that there is an exact differential of what we call a new global T-coordinate. This means that T is a differentiable function of global rain coordinates. This coordinate $T = t + f(r)$ is global because (a) Schwarzschild's t and r (along with ϕ) already label every event for $r > 0$, and (b) Schwarzschild coordinates and metric satisfy Einstein's equations. From (20) and (21):

$$dT = dt + d[f(r)] = dt + \left(\frac{2M}{r}\right)^{1/2} \left(1 - \frac{2M}{r}\right)^{-1} dr \tag{22}$$

To validate the new global rain coordinate T, we need to express it in the already-validated Schwarzschild coordinates. To start this process integrate (21):

$$f(r) \equiv \int_0^r \left(\frac{2M}{r}\right)^{1/2} \left(1 - \frac{2M}{r}\right)^{-1} dr = -(2M)^{1/2} \int_0^r \frac{r^{1/2} dr}{2M - r} \tag{23}$$

A table of integrals helps us to integrate the right side of (23). The result is:

$$f(r) = 4M \left(\frac{r}{2M}\right)^{1/2} - 2M \ln \left| \frac{1 + (2M/r)^{1/2}}{1 - (2M/r)^{1/2}} \right| \tag{24}$$

Finally, from (22) and (24):

$$\boxed{T = t + f(r) = t + 4M \left(\frac{r}{2M}\right)^{1/2} - 2M \ln \left| \frac{1 + (2M/r)^{1/2}}{1 - (2M/r)^{1/2}} \right|} \tag{25}$$

where we have arbitrarily set some constants of integration equal to zero. Equation (25) is the final proof that the new global rain T is valid, since it transforms directly from valid Schwarzschild t and r global coordinates.

?

Objection 6. *I still think that your definition of T derived from a local coordinate increment $\Delta t_{\rm rain}$ in equations (22) through (25) violates your own prohibition in Section 5.7 of a local-to-global transformation.*

!

No, we do *not* transform from local rain coordinates to global rain coordinates, which would be illegal. Instead, we start with the global Schwarzschild expression for $\Delta t_{\rm rain}$ in (16), take its differential limit in (20), which converts it to the global rain differential dT. The key step in that conversion is to recognize that equation (20) contains an exact differential in global Schwarzschild coordinates. Note that this does not happen for shell time in Schwarzschild coordinates:

$$\lim_{\Delta t \to 0} \Delta t_{\rm shell} \to \left(1 - \frac{2M}{r}\right)^{1/2} dt \qquad (26)$$

The right side of (26) cannot be expressed as $d[T(r,t)]$, the exact differential of a coordinate $T(r,t)$.

The exact differential in (20) allows us to complete the derivation and validation of the global rain T-coordinate in equation (25). Is this magic? No, but it does require sophisticated use of calculus.

QUERY 4. Differentiate T (*Optional*)
Take the differential of (25) to show that the result is (22).

Coordinate transformation to global rain coordinates

We worked in Schwarzschild coordinates to find a function $T(t,r) = t + f(r)$ whose differential matches local rain frame time. We can free this T of its origin and regard it as a new label for each spacetime event. But after that addition there is no need to use both T and t to label events. We can now eliminate t from the list, because we can always find it if we know the value of T (along with r) through $t = T - f(r)$. So we perform the **coordinate transformation** (25), in which we replace one set of global coordinates (t, r, ϕ) with a new set of global coordinates (T, r, ϕ). We call the result **global rain coordinates**—or historically, **Painlevé-Gullstrand coordinates**.

Payoff: "Intuitive" global rain metric

Why go to all this bother? In order to derive (in the next section) a global rain metric that describes the motion of a stone or light flash across the event horizon in a manner more comfortable to our intuition. The global rain metric encourages our unlimited, free exploration of *all* spacetime outside, at, and inside the event horizon.

An immediate payoff of global rain coordinates is local shell coordinates expressed in global rain coordinates (Box 2).

Comment 4. Global T is defined everywhere.
Although we started from local shell coordinates which exist only for $r > 2M$, our new T coordinate is defined everywhere, even at and inside the event

Box 2. Local shell coordinates expressed in global rain coordinates

We now derive local shell coordinates as functions of global rain coordinates. Equation (11) gives Δt_{shell} as a function of the Schwarzschild t-coordinate increment:

$$\Delta t_{\text{shell}} \equiv \left(1 - \frac{2M}{\bar{r}}\right)^{1/2} \Delta t \qquad (27)$$

The Schwarzschild t-coordinate does increase without limit along the worldline of a stone that approaches $r = 2M$, but shells exist only outside this event horizon, where (27) is well-behaved. Write equation (22) in approximate form:

$$\Delta t \approx \Delta T - \left(\frac{2M}{\bar{r}}\right)^{1/2} \left(1 - \frac{2M}{\bar{r}}\right)^{-1} \Delta r \qquad (28)$$

Substitute (28) into (27) to yield:

$$\boxed{\begin{aligned} \Delta t_{\text{shell}} \equiv &\left(1 - \frac{2M}{\bar{r}}\right)^{1/2} \Delta T \qquad (29) \\ &- \left(\frac{2M}{\bar{r}}\right)^{1/2} \left(1 - \frac{2M}{\bar{r}}\right)^{-1/2} \Delta r \end{aligned}}$$

Equations for Δy_{shell} and Δx_{shell} do not depend on Δt, so we copy them directly from (12) and (13).

$$\boxed{\Delta y_{\text{shell}} \equiv \left(1 - \frac{2M}{\bar{r}}\right)^{-1/2} \Delta r \qquad (30)}$$

$$\boxed{\Delta x_{\text{shell}} \equiv \bar{r}\Delta\phi \qquad (31)}$$

Note: Expressions for Δt_{shell} and Δy_{shell} are real only for $r > 2M$, consistent with the conclusion that no shell can exist inside the event horizon.

horizon, because its defining equation (25) contains only global Schwarzschild coordinates, which span all of spacetime.

Should we worry that the last term on the right side of (25) blows up as $r \to 2M$? No, because Schwarzschild t in (25) blows up in the opposite direction in such a way that T is continuous across $r = 2M$. Our only legitimate worry is continuity of the resulting metric, which is equation (32). Section 7.5 shows this metric to be continuous across $r = 2M$.

7.5 ■ THE GLOBAL RAIN METRIC

Move inward across the event horizon.

Section 7.4 created a global T-coordinate, validated it by direct transformation from already-approved global Schwarzschild coordinates, and installed it in a new set of global rain coordinates. To derive the **global rain metric**, solve the dT-transformation (22) for dt, substitute the result into the Schwarzschild metric, and collect terms to yield:

$$\boxed{\begin{gathered} d\tau^2 = \left(1 - \frac{2M}{r}\right) dT^2 - 2\left(\frac{2M}{r}\right)^{1/2} dT\,dr - dr^2 - r^2 d\phi^2 \qquad (32) \\ -\infty < T < +\infty, \quad 0 < r < \infty, \quad 0 \le \phi < 2\pi \quad \text{(global rain metric)} \end{gathered}}$$

Metric (32), with its connectedness (topology), provides a complete description of spacetime around a non-spinning black hole, just as the

Schwarzschild metric does. In addition, all Schwarzschild-based difficulties with worldlines that cross the event horizon disappear.

QUERY 5. Global rain metric

Substitute dt from (22) into the Schwarzschild metric to verify the global rain metric (32).

QUERY 6. Flat spacetime far from the black hole

Show that as $r \to \infty$ metric (32) becomes the metric for flat spacetime in global coordinates T, r, ϕ.

Use global rain metric from now on.

Comment 5. USE THE GLOBAL RAIN METRIC FROM NOW ON.
From now on in this book we use the global rain metric—and expressions derived from it—to analyze events in the vicinity of the non-spinning black hole.

Principle of General Covariance

Question: When all is said and done, which set of coordinates is the "correct" one for the non-spinning black hole: Schwarzschild coordinates or rain coordinates or some other set of global coordinates? *Answer: Every global coordinate system is valid provided it is either (a) submitted to Einstein's equations, which return a global metric or (b) transformed from an already-validated global coordinate system.* This reflects a fundamental principle of general relativity with the awkward technical name **Principle of General Covariance**. In this book we repeat over and over again that no single observer measures map coordinates directly (back cover). To overlook the Principle of General Covariance by attaching physical meaning to global coordinates is wrong and sets oneself up to make fundamental errors in the predictions of general relativity.

EVERY GENERATION MUST LEARN ANEW

One of the fundamental principles of general relativity (the principle of general covariance) states that all [global] spacetime coordinate systems are equally valid for the description of nature and that metrics that are related by a coordinate transformation are physically equivalent. This principle sounds simple enough, but one repeatedly finds in the literature arguments that amount to advocacy for the interpretation based on one set of [global] coordinates to the exclusion of the interpretation that is natural when using another set of [global] coordinates for the same set of events. It has been said that each generation of physicists must learn anew (usually the hard way) the meaning of Einstein's postulate of general covariance.

—Richard C. Cook and M. Shane Burns

Box 3. An Unlimited Number of Global Coordinate Systems

General relativity uses two methods to derive a global metric: *Method 1:* Choose an arbitrary set of global coordinates and submit them to Einstein's equations, which return the global metric expressed in those coordinates. Karl Schwarzschild did this to derive the Schwarzschild metric. *Method 2:* Transform an already-validated set of global coordinates to another set, then substitute the new coordinates into the already-verified metric of Method 1 to yield a metric in the new global coordinates. In the present chapter we use Method 2 to go from the Schwarzschild metric to the global rain metric.

How many different global coordinate systems are there for the non-spinning black hole? An unlimited number! Any one-to-one transformation from one valid set of global coordinates to another set of global coordinates makes the second set valid as well, provided it meets the usual criteria of uniqueness and smoothness (Section 5.8).

As a simple case, make the transformation:

$$T^\dagger \equiv KT \tag{33}$$

where K is any real number—or even a simple minus sign! With this substitution, global rain metric (32) becomes

$$d\tau^2 = \left(1 - \frac{2M}{r}\right)\left(\frac{dT^\dagger}{K}\right)^2 - 2\left(\frac{2M}{r}\right)^{1/2}\frac{dT^\dagger}{K}dr - dr^2 - r^2 d\phi^2 \tag{34}$$

If K is negative, then global $T^\dagger$ runs backward along the worldline of every stone. This does not bother us; Schwarzschild t does the same along some worldlines (Figure 2). Equations for local shell and local rain coordinates are similarly modified, which does not change any prediction or the result of any measurement in these local frames.

In a similar manner we can let $r^\dagger \equiv Qr$ and/or $\phi^\dagger \equiv -\phi$. You can write down new metrics with any one or any pair of these new coordinates. These changes may seem trivial, but they are not. For example, choose $K = Q = M^{-1}$. The result is the variables T/M and r/M, so-called **unitless coordinates**, in which curves are plotted in many figures of this book. Plots with unitless coordinates are correct for *every* non-spinning black hole, independent of its mass M.

Result: Global rain metric (32) is only one of an unlimited set of alternative, equally-valid global metrics for the non-spinning black hole, each one expressed in a different global coordinate system. The examples in this box may be no more useful than the original global rain metric, but there are other global metrics that highlight some special property of the non-spinning black hole. Look up **Eddington-Finkelstein coordinates** and **Kruskal-Szekeres coordinates** and their metrics.

In particular: Fixation on an interpretation based on one set of arbitrary coordinates can lead to the mistaken belief that global coordinate differences correspond to measurable quantities (Section 2.7).

Reminder of Einstein's error

Comment 6. You know some relativity that Einstein missed!
Chapter 5 says that Einstein took seven years to appreciate that global coordinate separations have no measurable meaning. But even then he did not fully understand this fundamental idea. At a Paris conference in 1922—seven years after he completed general relativity—Einstein worried about what would happen at a location where the denominator of the dr^2 term in the Schwarzschild metric, namely $(1 - 2M/r)$, goes to zero. He said it would be "an unimaginable disaster for the theory; and it is very difficult to say *a priori* what would occur physically, because the theory would cease to apply." (In 1933 the Belgian priest Georges Lemaître recognized that the apparent singularity in Schwarzschild coordinates at $r = 2M$ is "fictional.") Einstein was also baffled by the $dTdr$ cross term in the global rain metric (32) presented to him by Paul Painlevé and rejected Painlevé's solution out of hand. This led to the eclipse of this metric for decades.

Congratulations: You know some relativity that Einstein did not understand!

QUERY 7. Map energy in global rain coordinates

A. Use the global rain metric and the Principle of Maximal Aging to derive the map energy of a stone in global rain coordinates. You can model the procedure on the derivation of the Schwarzschild-coordinate expression for E/m in Section 6.2, but will need to alter some of the notation. Demonstrate this result:

$$\boxed{\frac{E}{m} = \left(1 - \frac{2M}{r}\right)\frac{dT}{d\tau} - \left(\frac{2M}{r}\right)^{1/2}\frac{dr}{d\tau} \qquad \text{(global rain coordinates)} \qquad (35)}$$

B. Substitute for dT from (22) and show that the result yields the Schwarzschild expression for map energy, equation (8) in Section 6.2.

C. Show that the map energy equation (35) applies to a stone in orbit around the black hole, not just to one that moves along the r-coordinate line.

D. Find an expression for dr/dT of the raindrop in rain map coordinates? Start with the first line of (5) and multiply both sides by dt/dT from (22). Show that the result is:

$$\frac{dr}{dT} = \frac{dr}{dt}\frac{dt}{dT} = -\left(\frac{2M}{r}\right)^{1/2} \qquad \text{(raindrop)} \qquad (36)$$

E. Show that the map worldline of a raindrop is given by the equation:

$$T - T_{\text{A}} = \frac{4M}{3}\left[\left(\frac{r_{\text{A}}}{2M}\right)^{3/2} - \left(\frac{r}{2M}\right)^{3/2}\right] \qquad \text{(raindrop)} \qquad (37)$$

where $(r_{\text{A}}, T_{\text{A}})$ locates the initial event on the plotted curve (Figure 3).

?

Objection 7. *Wait! The right side of (37) is identical to the right side of (2). The two equations are both correct if and only if the left sides are equal:*

$$T - T_{\text{A}} = \tau_{\text{raindrop}}[r_{\text{A}} \to r] \qquad (38)$$

But these two quantities are completely different—apples and oranges! The left side is the difference in a global coordinate, while the right side is the lapse of wristwatch time of a particular falling stone.

!

For our own convenience, we *chose* global rain coordinates so that equation (38) is valid. This is playing with fire, of course, because it is dangerous to assume that any given global coordinate corresponds to a measurable quantity (Section 5.8). But we are adults now, able to see the pitfalls of mature life. The goal, as always, is correct prediction of measurements and observations.

Raindrop equation (36) for dr/dT looks quite different from raindrop equation (5) for dr/dt, but the two must predict the same raindrop wristwatch time from r_{A} to a smaller r-coordinate given by (2). Let's check this: For a raindrop, set $E/m = 1$ in (35) and multiply through by $d\tau_{\mathrm{raindrop}}$:

$$d\tau_{\mathrm{raindrop}} = \left(1 - \frac{2M}{r}\right) dT - \left(\frac{2M}{r}\right)^{1/2} dr \tag{39}$$

Solve (36) for dT and substitute into (39).

$$\begin{aligned} d\tau_{\mathrm{raindrop}} &= -\left(1 - \frac{2M}{r}\right)\left(\frac{2M}{r}\right)^{-1/2} dr - \left(\frac{2M}{r}\right)^{1/2} dr \\ &= -\left(\frac{r}{2M}\right)^{1/2} dr \end{aligned} \tag{40}$$

which is the same as equation (1), so its integral from r_{A} to r must be (2), as required. This is an example of an important property of different global metrics: *Every global metric must predict the same result of a given measurement or observation* (Objection 5).

Box 4 carries out a Lorentz transformation from local shell coordinates in Box 2 to local rain coordinates, then verifies that local rain coordinates are valid everywhere, not just outside the event horizon.

Comment 7. An observer passes through a sequence of local frames. The rain observer rides on a raindrop (Definition 4, Section 7.7). In curved spacetime, local inertial frames are limited in both space and time. During her fall, the rain observer passes through a series of local rain frames as shown in Figure 3. Equations (42) through (44) contain an $\bar{r}$, assumed to have the same value everywhere in that local frame. Although absent from the equations, similar average $\bar{T}$ and $\bar{\phi}$ are implied by all three local rain coordinate equations. *Result:* Each rain observer passes through a sequence of local inertial frames. Similar statements also apply to, and may seem more natural for, local shell frames with local coordinates (29) through (31).

Box 4 derives *local* rain coordinates expressed in *global* rain coordinates. This simplifies local rain coordinates compared with those expressed in Schwarzschild coordinates in equations (16) through (19).

Figure 3 displays several rain observer worldlines on the $[r, T]$ slice. We surround one worldline with a worldtube—shown in cross section—that contains local rain frames through which this rain observer passes in sequence. With $\Delta t_{\mathrm{rain}} = \Delta T = 0$, equation (43) tells us that Δy_{rain} coordinate lines are horizontal in this figure. Finally, Δx_{rain} coordinate lines, which are perpendicular to both Δt_{rain} and Δy_{rain} coordinate lines, project outward, perpendicular to the page in Figure 3.

We could cover the worldtube in Figure 3 with adjacent or overlapping local rain frames. The resulting figure would be analogous to Figure 5 in Section 2.2, which places overlapping local flat maps along the spatial path from Amsterdam to Vladivostok along Earth's curved surface.

Box 4. Local rain coordinates expressed in global rain coordinates

Apply the Lorentz transformation to local shell coordinates in Box 2 to derive local rain coordinates. Relative velocity in the Lorentz transformation lies along the common Δy_{shell} and Δy_{rain} line. With this change, Lorentz transformation equations of Section 1.10 become:

$$\begin{aligned}\Delta t_{\text{rain}} &= \gamma_{\text{rel}}\left(\Delta t_{\text{shell}} - v_{\text{rel}}\Delta y_{\text{shell}}\right) \\ \Delta y_{\text{rain}} &= \gamma_{\text{rel}}\left(\Delta y_{\text{shell}} - v_{\text{rel}}\Delta t_{\text{shell}}\right) \\ \Delta x_{\text{rain}} &= \Delta x_{\text{shell}}\end{aligned} \tag{41}$$

Substitute v_{rel} and γ_{rel} from (14), along with local shell coordinates from Box 2, into equations (41) to obtain expressions for local rain coordinates as functions of global rain coordinates:

$$\boxed{\Delta t_{\text{rain}} \equiv \Delta T} \tag{42}$$

$$\boxed{\Delta y_{\text{rain}} \equiv \Delta r + \left(\frac{2M}{\bar{r}}\right)^{1/2}\Delta T} \tag{43}$$

$$\boxed{\Delta x_{\text{rain}} \equiv \bar{r}\Delta\phi} \tag{44}$$

Coefficients on the right sides of these equations remain real inside the event horizon, so local rain coordinates are valid there (Figure 3).

Wait! How can we justify our derivation of rain coordinates from local shell coordinates, which are valid only outside the event horizon? To do so, we need to show that local rain coordinates lead back to the global rain metric, which is valid everywhere outside the singularity.

$$\Delta\tau^2 \approx \Delta t_{\text{rain}}^2 - \Delta y_{\text{rain}}^2 - \Delta x_{\text{rain}}^2 \tag{45}$$

Substitute into (45) from (42) through (44):

$$\begin{aligned}\Delta\tau^2 \approx \Delta T^2 - &\left[\Delta r + \left(\frac{2M}{\bar{r}}\right)^{1/2}\Delta T\right]^2 \\ &- \bar{r}^2\Delta\phi^2\end{aligned} \tag{46}$$

Multiply out:

$$\begin{aligned}\Delta\tau^2 \approx &\left(1 - \frac{2M}{\bar{r}}\right)\Delta T^2 \\ &- 2\left(\frac{2M}{\bar{r}}\right)^{1/2}\Delta T\Delta r - \Delta r^2 - \bar{r}^2\Delta\phi^2\end{aligned} \tag{47}$$

In the calculus limit, equation (47) becomes the global rain metric (32). The global rain metric is valid everywhere down to the singularity; therefore local rain coordinates can be constructed down to the singularity.

Box 5 derives the global rain embedding diagram, Figure 4, from the global rain metric (32) and compares it with the embedding diagram for Schwarzschild coordinates.

7.6 ■ TETRAD FORMS OF THE GLOBAL RAIN METRIC

A difference of squares hides the cross term.

Global rain metric (32) has a cross term. The metric for any local inertial frame derived from this global metric does not have a cross term. For example:

$$\Delta\tau^2 \approx \Delta t_{\text{shell}}^2 - \Delta y_{\text{shell}}^2 - \Delta x_{\text{shell}}^2 \tag{49}$$

$$\Delta\tau^2 \approx \Delta t_{\text{rain}}^2 - \Delta y_{\text{rain}}^2 - \Delta x_{\text{rain}}^2 \tag{50}$$

Why this difference between global and local metrics? Can we use a set of local inertial coordinates to create a form of the global metric that consists of the sum and difference of squares? Try it! From expressions for local shell and

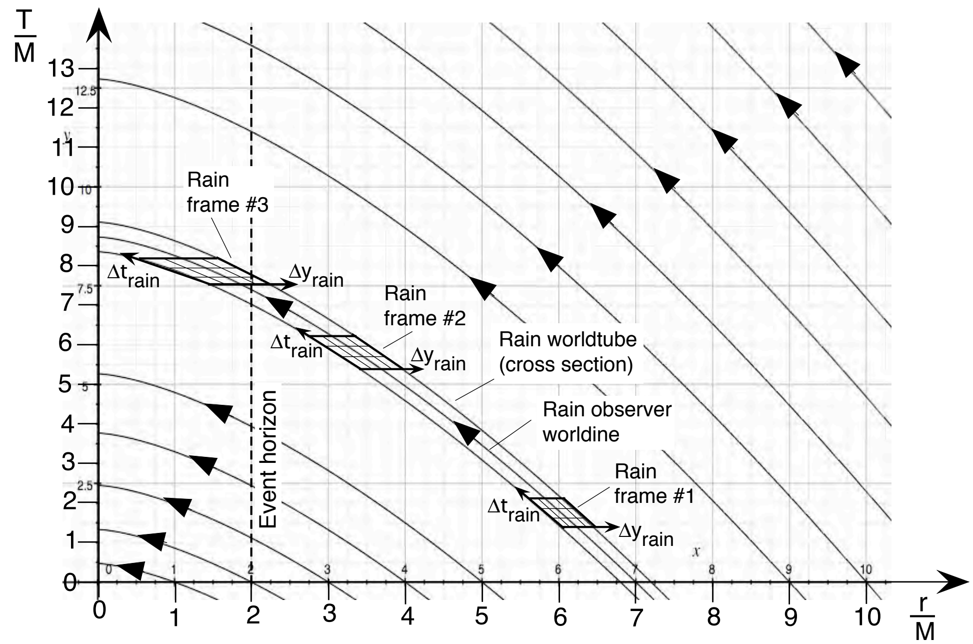

FIGURE 3 Raindrop worldlines plotted on an $[r, T]$ slice. Note that these worldlines are continuous through the event horizon (compare Figure 2). All these worldlines have the same shape and are simply displaced vertically with respect to one another. Around one of these worldlines we construct a worldtube (shown in cross section on this slice) that bounds local rain frames through which that rain observer passes.

rain coordinates in Boxes 2 and 4, respectively, simply write down two differential forms of the global metric. From local shell coordinates (Box 2):

$$d\tau^2 = \left[\left(1-\frac{2M}{r}\right)^{1/2} dT - \left(\frac{2M}{r}\right)^{1/2}\left(1-\frac{2M}{r}\right)^{-1/2} dr\right]^2 \tag{51}$$

$$-\left(1-\frac{2M}{r}\right)^{-1} dr^2 - r^2 d\phi^2 \qquad \text{(global rain metric)}$$

$$-\infty < T < +\infty, \quad 0 < r < \infty, \quad 0 \le \phi \le 2\pi$$

And from local rain coordinates (Box 4):

$$d\tau^2 = dT^2 - \left[dr + \left(\frac{2M}{r}\right)^{1/2} dT\right]^2 - r^2 d\phi^2 \qquad \text{(global rain metric)} \tag{52}$$

$$-\infty < T < +\infty, \quad 0 < r < \infty, \quad 0 \le \phi \le 2\pi$$

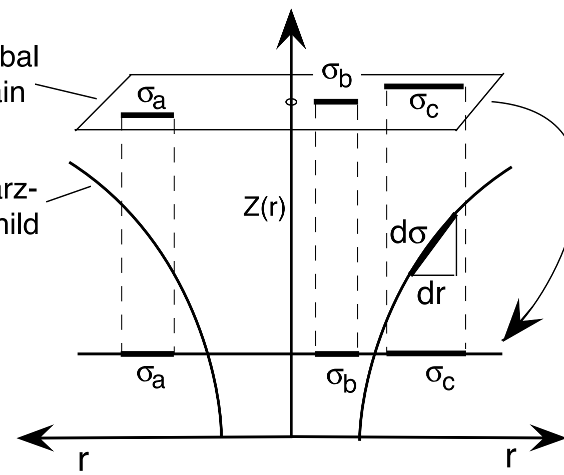

FIGURE 4 Figure for Box 5. Compare the embedding diagram outside the event horizon for Schwarzschild coordinates (the funnel) in Figures 11 through 13 in Section 3.9 with the flat embedding diagram of global rain coordinates (the flat surface shown in perspective across the top). The function $Z(r)$ is the fictional dimension we add in order to visualize these surfaces.

Box 5. Embedding diagrams for Schwarzschild and global rain coordinates.

Set $dT = 0$ in global rain metric (32), which then retains terms that include only r and ϕ:

$$d\sigma^2 = ds^2 = dr^2 + r^2 d\phi^2 \qquad (dT = 0) \qquad (48)$$

Surprise! The differential ruler distance $d\sigma$ obeys Euclidean flat-space geometry, which leads to the flat embedding diagram at the top of Figure 4 (point at $r = 0$ excluded). Because the global rain embedding diagram is flat, we can simply sum increments $d\sigma$ to draw arbitrary lines or curves with measured lengths σ_{a}, σ_{b}, and σ_{c} (for simplicity, drawn as parallel straight lines in Figure 4).

Figure 4 also repeats the embedding diagram outside the event horizon in Schwarzschild global coordinates from Figures 11 through 13 in Section 3.9.

Outside the event horizon both embedding diagrams are valid for what they describe. And the flat global rain embedding diagram is valid inside the event horizon as well.

What's going on here? Is space flat or funnel-shaped around this black hole? *That depends on our choice of global coordinates! Spacetime* as a unity is curved; but Nature does not care how we share the description of spacetime curvature among the terms of the global metric. In global rain coordinates the dT^2 and $dTdr$ terms describe spacetime curvature, which leaves the $[r, \phi]$ embedding diagram to show flat space. In contrast, for the Schwarzschild metric the dt^2 term and the dr^2 term share the description of spacetime curvature, which yields a funnel outside the event horizon on the $[r, \phi]$ embedding diagram. Each embedding diagram and global light cone diagram is a child of our (arbitrary!) choice of global coordinates in which they are expressed.

Recall Herman Minkowski's declaration (Section 2.7): "Henceforth space by itself and time by itself are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality." Each global metric displays that union in a different way.

Are (51) and (52) valid global metrics? Yes! In Query 8 you multiply out these global metrics to show that they are algebraically equivalent to the original global rain metric (32).

QUERY 8. The same global metric
Expand the right side of (51) and the right side of (52). Show that in both cases the result is equal to the right side of the original global rain metric (32).

Conclusion: All three forms of the global rain metric, (32), (51), and (52) are simply algebraic rearrangements of one another. So what? Why bother? Here's why: Suppose we are first given either metric (51) or metric (52). In that case we can immediately write down the expressions Δt_{frame}, Δy_{frame}, and Δx_{frame} for *some* local inertial frame. (We may not know right away *which* local inertial frame it is.)

A metric of the form (51) or (52) is called the **tetrad form** of the global metric. "Tetra" means "four," in this case the four dimensions of spacetime. A tetrad form rearranges the global metric in a form more useful to us.

DEFINITION 1. A tetrad form of a global metric
A **tetrad form of a global metric** consists of a sum and difference of squares, with no additional terms.

DEFINITION 2. Tetrad
A **tetrad** is a set of four differential expressions, each of which is squared in a "tetrad form" of a global metric.

Example: Equation (52) is the global rain metric in the tetrad form that leads to the local rain frame coordinates in Box 4.

?

Objection 8. *You said the tetrad consists of* four *differential expressions. I see only three.*

!

You're right. In most of this book we use only two global space dimensions, those on a slice through the center of the black hole. A third global space dimension would add a fourth Δz_{rain} component to the tetrad. We retain the professional terminology *tetrad* in spite of our simplification.

Comment 8. Tetrad as link
A tetrad is the link between a global metric and a local inertial frame. To specify a particular tetrad, give both the local inertial frame and the global metric. The local inertial frame stretches differentials d in the global metric to deltas Δ in the local inertial frame. This stretch creates elbow room to make measurements in the local inertial frame.

The global metric in tetrad form immediately translates into expressions for local inertial coordinates (Box 6). The remainder of this book will make primary use of such global tetrad metrics.

Box 6. A Brief History of Tetrads

The concept of a tetrad originates from the "repère mobile" (*moving frame*), introduced by Élie Cartan in the 1930s. Cartan showed that a sequence of local inertial coordinate systems, grouped along a set of curves such as worldlines, can provide a complete global description of a curved spacetime. He did so by introducing new calculus concepts on curved spaces, which extended the foundations of differential geometry laid by Bernhard Riemann in 1854.

Cartan's moving-frame theory was incomprehensible to Einstein, but later physicists found it useful and even necessary to study elementary particle physics in curved spacetime.

In this book we simplify the moving frame, or tetrad, to its most basic element: a set of local inertial frames in which motion is described using local inertial coordinates, with each frame and each local coordinate system related to the global coordinates provided by the metric. The tetrad is the bridge between the global metric and the local inertial metric in which we carry out all measurements and observations.

Because this metric is the sum and difference of the squares of the tetrad—such as in equations (51) and (52)—general relativists sometimes call the tetrad the **square root of the metric.**

7.7 ■ RAIN WORLDLINES OF LIGHT

The light flash is ingoing or outgoing—or "outgoing."

In the present chapter we consider only r-motions, motions that can be either ingoing or outgoing along r-coordinate lines. (Chapter 11 analyzes the general motion of light in global coordinates.) The r-motion of light is easily derived from global rain metric (53), which you show in Query 8 to be equivalent to global rain metric (32).

QUERY 9. Light r-motion in global rain coordinates

A. Multiply out the right side of (53) to show that it is equivalent to the global rain metric (32):

$$\boxed{d\tau^2 = -\left[dr + \left\{1 + \left(\frac{2M}{r}\right)^{1/2}\right\} dT\right]\left[dr - \left\{1 - \left(\frac{2M}{r}\right)^{1/2}\right\} dT\right] - r^2 d\phi^2 \qquad (53)}$$

(global rain metric)

B. For light ($d\tau = 0$) that moves along the r-coordinate line ($d\phi = 0$), show that (53) has two solutions for the rain map velocity of light, which are summarized by equation (54):

$$\frac{dr}{dT} = -\left(\frac{2M}{r}\right)^{1/2} \pm 1 \qquad \text{(light flash that moves along the } r\text{-coordinate line)} \qquad (54)$$

$(- =$ incoming light; $+ =$ outgoing or "outgoing" light)

C. Look separately at each element of the plus-or-minus sign in (54). Show that the solution with the lower sign $(-)$ describes an *incoming* flash (light moving inward along the r-coordinate line). Then show that the solution with the upper sign $(+)$ describes an *outgoing* flash (light moving outward along the r-coordinate line) outside the event horizon, but an "*outgoing*" flash inside the event horizon (Definition 3).

DEFINITION 3. "Outgoing" light flash.

The **"outgoing" light flash**—with quotes—is a flash inside the event horizon whose r-motion is described by (54) with the plus sign. Inside the event horizon dr is negative as T advances (positive dT), so that even the "outgoing" light flash moves to smaller r-coordinate. The light cone diagram Figure 5 shows this, and Figure 6 displays longer worldlines of light flashes emitted sequentially by a plunging raindrop. Compare the global rain worldline in Figure 6 with the Schwarzschild worldlines in Figure 2.

We want to plot worldlines of incoming and outgoing (and "outgoing") light flashes. Rewrite (54) to read

$$dT = \frac{dr}{-\left(\dfrac{2M}{r}\right)^{1/2} \pm 1} = \frac{r^{1/2}dr}{-(2M)^{1/2} \pm r^{1/2}} \qquad (r\text{-moving flash}) \qquad (55)$$

We carry the plus-or-minus sign along as we integrate to find expressions for light that moves in either r-direction. Make the substitution:

$$u \equiv -(2M)^{1/2} \pm r^{1/2} \qquad (56)$$

From (56),

$$r^{1/2} = \pm[u + (2M)^{1/2}] \quad \text{and} \quad dr = +2\left[u + (2M)^{1/2}\right] du \qquad (57)$$

With these substitutions, equation (55) becomes

$$\begin{aligned} dT &= \pm 2\frac{\left[u + (2M)^{1/2}\right]^2 du}{u} \qquad (r\text{-moving flash}) \qquad (58)\\ &= \pm 2\left[u + 2\,(2M)^{1/2} + \frac{2M}{u}\right] du \end{aligned}$$

Integrate the second line of (58) from an initial u_0 to a final u. The result is:

$$\pm(T - T_0) = u^2 - u_0^2 + 4(2M)^{1/2}\,(u - u_0) + 4M \ln\left|\frac{u}{u_0}\right| \quad (r\text{-moving flash}) \quad (59)$$

To restore global rain coordinates in (59), reverse the substitution in (56). (*Hint:* To save time—and your sanity—replace $\pm$ in (59) with a symbol such as Q.) There are two cases: *First case:* an incoming flash from a larger r-coordinate $r_0 = r_{\mathrm{A}}$ at coordinate $T_0 = T_{\mathrm{A}}$ to a lower r at coordinate T. For

Incoming flash

this incoming flash, take the lower minus signs in (56) and (59). Multiply both sides of the result by minus one to obtain:

$$T - T_{\mathrm{A}} = (r_{\mathrm{A}} - r) - 4M\left[\left(\frac{r_{\mathrm{A}}}{2M}\right)^{1/2} - \left(\frac{r}{2M}\right)^{1/2}\right] \qquad (60)$$
$$+ 4M \ln\left[\frac{1 + (r_{\mathrm{A}}/2M)^{1/2}}{1 + (r/2M)^{1/2}}\right] \quad \text{(incoming flash)}$$

QUERY 10. Horizon-to-crunch global T-coordinate lapse for light

A. Verify that for $r = r_{\mathrm{A}}$ in (60), the elapsed global T-coordinate $T - T_{\mathrm{A}}$ is zero, as it must be for light.

B. Show that from the event horizon $r_{\mathrm{A}} = 2M$ to the crunch point $r = 0$, the elapsed T-coordinate $T - T_{\mathrm{A}} = 0.773M$.

C. Compare the result of Item B with the event-horizon-to-crunch wristwatch time $\tau_{\mathrm{raindrop}} = (4/3)M$ in equation (3). Why is the result for a light flash in (60) less than the result for the raindrop? Are the plots in Figure 6 consistent with this inequality?

Outgoing flash

Second case: the outgoing and "outgoing" light flashes from an initial r-coordinate $r_0 = r_{\mathrm{L}}$ at coordinate T_{L} to a final r at coordinate T. In this case we must take the upper, plus signs in (56) and (59). The result is:

$$T - T_{\mathrm{L}} = (r - r_{\mathrm{L}}) + 4M\left[\left(\frac{r}{2M}\right)^{1/2} - \left(\frac{r_{\mathrm{L}}}{2M}\right)^{1/2}\right] \qquad (61)$$
$$+ 4M \ln\left[\frac{1 - (r/2M)^{1/2}}{1 - (r_{\mathrm{L}}/2M)^{1/2}}\right] \quad \text{(outgoing flash)}$$

Inside event horizon, "outward" means worldline with $dr < 0$.

What does equation (61) predict when $r_{\mathrm{L}} < 2M$? Worldlines of light that sprout upward from the raindrop worldline in Figure 6 show that after the diver falls through the event horizon, even the "outward" flash moves to smaller r in global rain coordinates.

Figure 5 uses equations (60) and (61) to plot light cones for a selection of events inside and outside of the event horizon.

QUERY 11. Motion to smaller r only

Use the dashed worldline of a stone in Figure 5 to explain, in one or two sentences, why "everything moves to smaller r-coordinate" inside the event horizon. *Hint:* Think of the connection between worldlines of stones and future light cones.

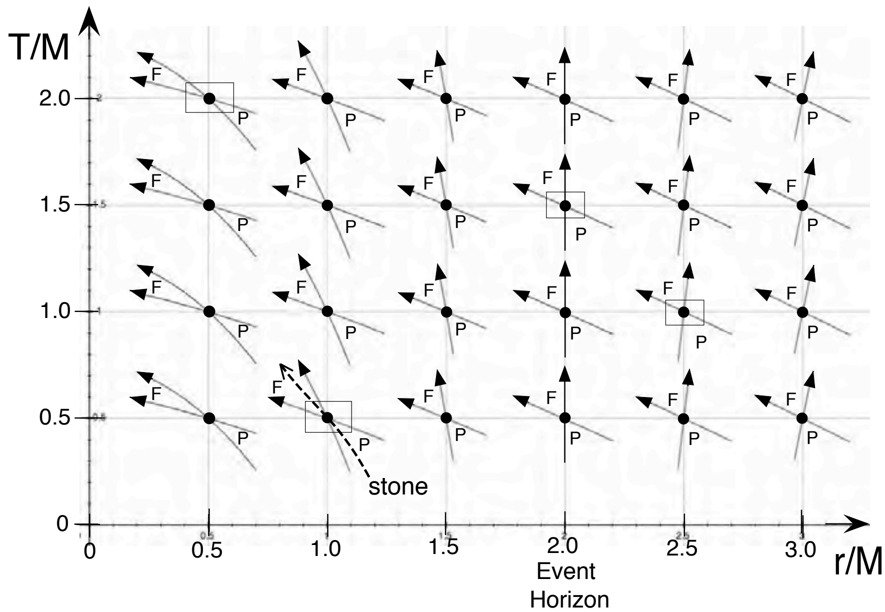

FIGURE 5 Light cone diagram on the $[r, T]$ slice, plotted from equations (60) and (61) for events both outside and inside the event horizon. Past and future events of each filled-dot-event are corralled inside the past (P) and future (F) light cone of that event. At each r-coordinate, the light cone can be moved up or down vertically without change of shape, as shown. Inside the event horizon, light and stones can move only to smaller r-coordinate. A few sample boxes show locally flat patches around a single event. The global rain T-coordinate conveniently runs forward along every worldline (in contrast to the Schwarzschild t-coordinate along some worldlines in the light cone diagram of Figure 8, Section 3.7).

QUERY 12. Detailed derivation *Optional*

Show details of the derivation of equations (60) and (61) from equation (59). Recall the hint that follows equation (59).

Time to celebrate! The raindrop worldline in Figure 6 is continuous and smooth as it moves inward across the horizon. Global rain coordinates yield predictions that are natural and intuitive for us. With global rain coordinates, we no longer need to reconcile the awkward contrast between the discontinuous Schwarzschild worldlines of Figure 2 and the smooth advance of raindrop wristwatch time in Figure 1 (even though both of these plots are valid and technically correct). The simplicity of results for global rain coordinates leads us to use them from now on to describe the non-spinning black hole. Farewell, Schwarzschild metric!

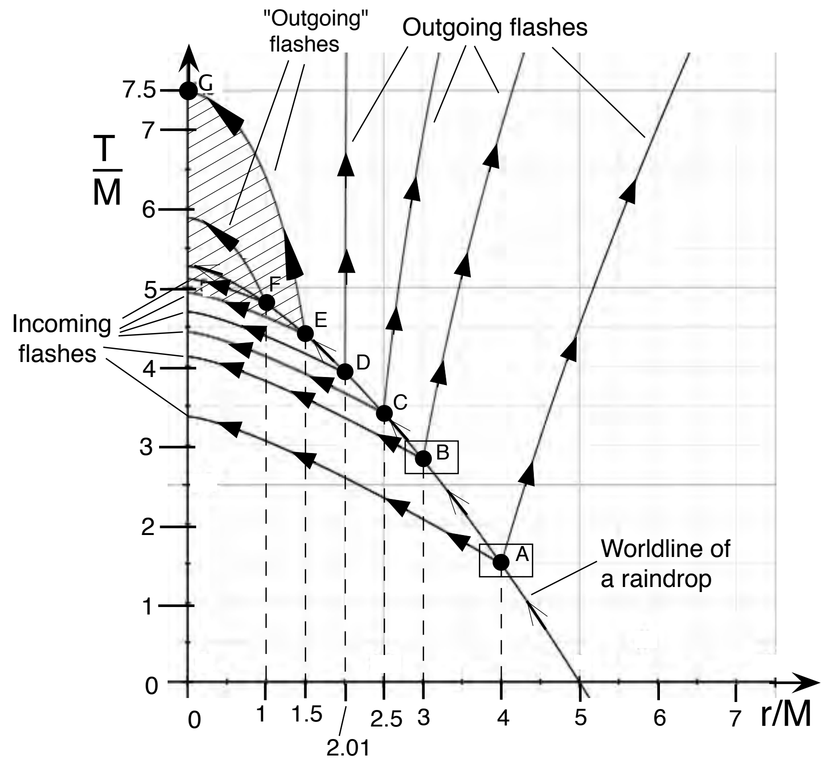

FIGURE 6 A raindrop passes $r/M = 5$ at $T/M = 0$ and thereafter emits both incoming and outgoing flashes at events A through F. "Outgoing" flashes—with quotes—from events E and F move to smaller global r-coordinates, along with everything else inside the event horizon. Little boxes at A and B represent two of the many locally flat patches through which the rain observer passes as she descends. When the rain diver reaches event E, her "range of possible influence" consists of events in the shaded region, for example event F.

7.8 ■ THE RAIN OBSERVER LOOKS—AND ACTS

Which distant events can the rain observer see? Which can she influence?

You ride a raindrop; in other words, you fall from initial rest far from the black hole. What do you see radially ahead of you? behind you? Of all events that occur along this r-line, which ones can you influence from where you are? Which of these events can influence you? When can you no longer influence any events? To answer these questions we give the raindrop some elbow room, turn her into a **rain observer** who makes measurements and observations in a series of local rain frames through which she falls. This definition specializes the earlier general definition of an observer (Definition 4, Section 5.7).

Definition: **Rain observer**

DEFINITION 4. Rain observer

A **rain observer** is a person or a data-collecting machine that rides a raindrop. As she descends, the rain observer makes a sequence of measurements, each measurement limited to a local inertial rain frame (Box 4).

Objection 9. *Wait: Go back! You have a fundamental problem that ruins everything. The global rain metric (32) contains the r-coordinate, but you have not defined the r-coordinate inside the event horizon. Section 3.3 defined the r-coordinate as "reduced circumference," that is, the circumference of a shell divided by 2π. But you cannot build a shell inside the event horizon, so you cannot define global coordinate r there. Therefore you have no way even to describe the worldline of the rain observer once she crosses the event horizon.*

Guilty as charged! Box 4 defined local rain coordinates and justified their validity inside the event horizon, but we have not formally defined the r-coordinate inside the event horizon, or how an observer might determine its value there. Here is one way (Box 7): As the rain observer drops from rest far from the black hole, she simultaneously releases a stone test particle from rest beside her and perpendicular to her direction of motion. Thereafter she uses radar or a meter stick to measure the distance to the stone. In this way she monitors her r-coordinate as she descends inside the event horizon.

Box 4 introduced local rain frames in which we can carry out and record measurements using special relativity. Small boxes in Figures 5 and 6 represent effectively flat patches on which we can construct local inertial frames. In this chapter we allow the rain observer to look only at events that lie before and behind her along her worldline. She can also send light flashes and stones to influence (as much as possible) this limited set of events. (Chapter 11 allows the raindrop observer to look all around her.)

QUERY 13. Observe ingoing and outgoing light flashes in a local rain frame. How do light flashes that we describe as ingoing, outgoing, and "outgoing" in global rain coordinates (Definition 3) move when observed entirely within a local rain frame? Answer this question with the following procedure or some other method.

A. From (42) and (43) show that:

$$\Delta y_{\text{rain}} = \left[\frac{\Delta r}{\Delta T} + \left(\frac{2M}{\bar{r}}\right)^{1/2}\right]\Delta t_{\text{rain}} \quad \text{(light flash that moves along the } r\text{-coordinate line)} \tag{66}$$

B. Use an approximate version of (54) to replace the square bracket expression in (66):

$$\frac{\Delta y_{\text{rain}}}{\Delta t_{\text{rain}}} = \pm 1 \qquad \text{(light flash that moves along the } r\text{-coordinate line)} \tag{67}$$

C. Is equation (67) a surprise—or obvious? What does each sign mean for measurement of light velocity inside a rain frame?

Box 7. Define the Value of r Inside the Event Horizon

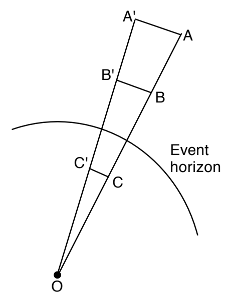

FIGURE 7 The rain observer measures her r-coordinate inside the event horizon.

Question: How can a rain observer inside the event horizon determine her current r-coordinate? *Answer:* To adapt the Chapter 3 definition of the r-coordinate—reduced circumference of the shell outside the event horizon—she measures only a tiny arc inside the event horizon.

The rain observer takes the path ABCO in Figure 7; the stone that accompanies her takes the converging path A'B'C'O. Draw a circular arc AA' and similar circular arcs BB' and CC'. The angle AOA' is the same for every arc, so the length of each arc represents the same fraction of the circumference of its corresponding circle. In equation form,

$$\frac{\begin{pmatrix}\text{length of}\\ \text{arc AA'}\end{pmatrix}}{\begin{pmatrix}\text{circumference}\\ \text{of shell thru A}\end{pmatrix}} = \frac{\begin{pmatrix}\text{length of}\\ \text{arc BB'}\end{pmatrix}}{\begin{pmatrix}\text{circumference}\\ \text{of shell thru B}\end{pmatrix}} \qquad (62)$$

The values of r-coordinates $r_{\rm A}$ and $r_{\rm B}$ are stamped on the shells outside the event horizon, so the denominators of the two sides of the equation become $2\pi r_{\rm A}$ and $2\pi r_{\rm B}$, respectively, and we cancel the common factor 2π.

If the angle at the center is small enough, we can replace the length of each circular arc with the straight-line distance measured between, say, A and A' shown in Figure 7. Call this measured distance AA'. And call BB' the corresponding straight-line distance measured between B and B'. Then (62) becomes,

$$\frac{\rm AA'}{r_{\rm A}} \approx \frac{BB'}{r_{\rm B}} \qquad (63)$$

The rain observer monitors the distance to her accompanying stone as she descends, with radar or—if the stone lies near enough—directly with a meter stick. While she is outside the event horizon, the rain observer reads the value of the r-coordinate $r_{\rm A}$ stamped on that spherical shell as she passes it and the measured distance AA' between the two rain frames, and later BB' as she passes and reads off $r_{\rm B}$. She verifies that this direct reading with the value of $r_{\rm B}$ is the same as that calculated with the equation:

$$\rm r_B \approx \frac{BB'}{AA'} r_A \qquad (64)$$

At any point C inside the event horizon, the observer measures distance CC' and *defines* her instantaneous r-coordinate $r_{\rm C}$ as:

$$r_{\rm C} \equiv \frac{\rm CC'}{\rm AA'} r_{\rm A} \qquad \text{(definition)} \qquad (65)$$

This definition of the r-coordinate inside the event horizon is a direct extension of its definition outside the event horizon and is valid for any observer falling along an r-coordinate line.

D. From observations inside a rain frame, is there any difference between a light flash we describe as *outgoing* and one we describe as *"outgoing"*? More generally, can observations carried out entirely inside a rain frame tell us whether that rain frame is outside of, at, or inside the event horizon?

"Range of possible influence"

As our rain observer arrives at any of the emission points A through F in Figure 6, she can try—by firing an ingoing or outgoing stone or light flash—to influence a later event located within the region embraced by the worldlines of the incoming and outgoing (or "outgoing") flashes from that event. The farther toward the singularity the rain observer falls, the smaller is this "range of possible influence." When she arrives at event E, for example, she can influence only events in the shaded region in Figure 6, including event F.

QUERY 14. Future events that the rain observer can still influence.
Make four photocopies of Figure 6. On each copy, choose one emission event A through D or F.

A. Shade the spacetime region in which a rain observer can influence future events once she has arrived at that emission event.
B. Which of these emission points is the last one from which the rain observer can influence events that occur at $r > 2M$?

As she crosses the event horizon, how long will it be on her wristwatch before she reaches the singularity? Equation (3) tells us this wristwatch time is $4M/3$ meters.

?

Objection 10. *Ha! I can live a lot longer inside the event horizon than your measly $4M/3$ meters of time. All I have to do, once I get inside the event horizon, is to turn on my rockets and boost myself radially outward. For example, I can fire super-powerful rockets at event E and follow the "outgoing" photon flash from E that reaches the singularity at Event G (top left corner of that figure). That final T-value is much greater than the T-value where the raindrop worldline reaches the singularity.*

!

Be careful! You want to maximize wristwatch time, not the span of global T which, remember, is usually *not* measureable time. The wristwatch time is zero along the worldline of a light flash, so the closer you come to that worldline the smaller will be your wristwatch time during descent from Event E to an event just below G in Figure 6.

?

Objection 11. *Okay, then! I'll give up the rocket blast, but I still want to know what is the longest possible wristwatch time for me to live after I cross the event horizon.*

!

Part B of Exercise 3 at the end of this chapter shows how to extend your lifetime to πM meters after you cross the event horizon, which is a bit longer than the raindrop $4M/3$ meters. You will show that the way to achieve this is to drip from the shell just outside the event horizon; that is, you release yourself from rest in global coordinates at $r = 2M^+$.

?

Objection 12. *Can I increase my lifetime inside the event horizon by blasting rockets in either ϕ direction to add a tangential component to my global velocity?*

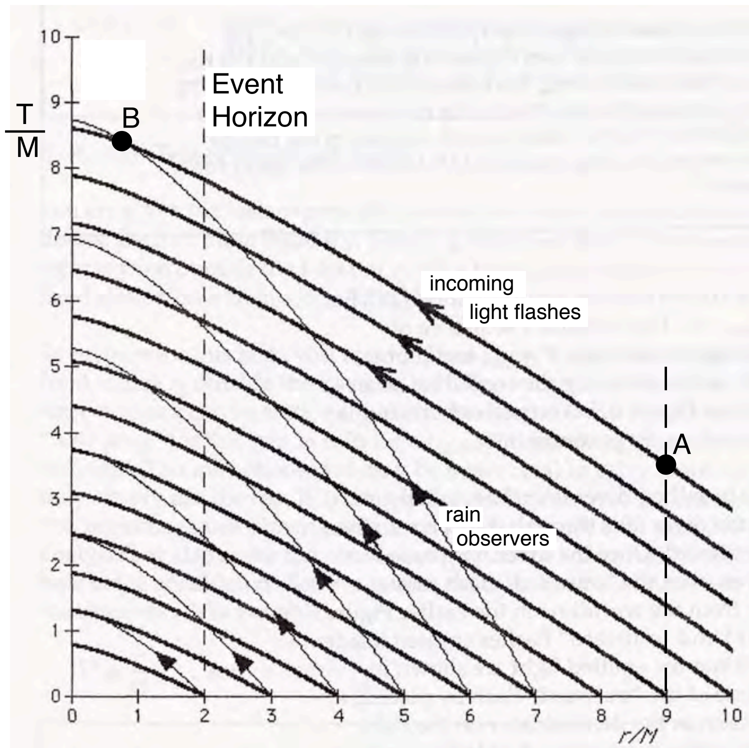

FIGURE 8 Worldlines of rain observers (thin curves) and incoming light flashes (thick curves) plotted on the $[r, T]$ slice. All rain observer worldlines have the same form and can be moved up and down without change in shape. Light-flash worldlines also have the same form. A shell observer at $r/M = 9$ emits a signal from Event A that a rain observer receives inside the event horizon at Event B.

!

The present chapter analyzes only r-motion. In the exercises of Chapter 8 you will show that the answer to your question is no; a tangential rocket blast *decreases* your lifetime inside the event horizon. A wristwatch time lapse of πM is the best you can do. Sorry.

Figure 8 displays both global worldlines of light flashes—thick curves derived from (60)—and worldlines of raindrops—thin curves derived from (2). It is evident from Figure 8 that news bulletins—incoming or outgoing electromagnetic radio bursts fired outside the event horizon—can be scheduled to catch up with the diver community at any predetermined r-coordinate.

QUERY 15. Can you see a rain diver ahead of you?
Compare Figures 6 and 8. Label as #1 the rain diver whose worldline is plotted in Figure 6. Label as #2 a second rain diver who falls along the *same* r-coordinate line in space as rain diver #1, but at a T-coordinate greater by ΔT. Use worldlines of Figure 8 to answer the following questions. (*Optional:*

Derive analytic solutions to these questions and compare the results with your answers derived from the figure.)

A. Over what range of delays ΔT will rain diver #2 be able to see the flash from Event E emitted by diver #1 but not the flash from Event F?

B. Over what range of delays ΔT will rain diver #2 be able to see the flash from Event D emitted by diver #1 but not the flash from Event E?

C. Answer this question decisively: Can any later diver #2 see a flash emitted by diver #1 at say, $r/M = 0.1$, just before diver #1 reaches the singularity?

QUERY 16. Can you see a rain diver behind you?
Extend the results of Query 15 to analyze a third rain diver labeled #3 who falls along the same r-coordinate line in space as the earlier two rain divers, but at a T-coordinate that is smaller by $-\Delta T$. Diver #3 looks outward at light pulses emitted by diver #1. Use the worldlines of Figure 6 to answer the following questions. (*Optional:* Derive analytic solutions to these questions and compare the results with your answers derived from the figure.)

A. Over what range of earlier launches $-\Delta T$ will rain diver #3 be able to see the flash from Event D emitted by diver #1 but not the flash from Event F?

B. Over what range of earlier launches $-\Delta T$ will rain diver #3 be able to see the flash from Event A emitted by diver #1 but not the flash from Event D?

C. Answer this question decisively: Can any earlier diver #3 see a flash emitted by diver #1 at say, $r/M = 0.1$, just before diver #1 reaches the singularity?

QUERY 17. Can you see the crunch point ahead of you?
You are the rain diver whose worldline is plotted in Figure 6. By some miracle, you survive to reach the center of the black hole. Show that you *cannot* see the singularity ahead of you before you arrive there. (What a disappointment after all the training, preparation, and sacrifice!)

QUERY 18. Rain frame energy
The rain frame is inertial. Therefore the expression for energy of a stone in rain frame coordinates is that of special relativity, namely $E_{\text{rain}}/m = dt_{\text{rain}}/d\tau$ (Section 1.7). Recall also from the differential version of (42) that $dt_{\text{rain}} = dT$.

A. Use (35) together with the special relativity expression for the rain frame energy of the stone and the above identification of global rain T with rain frame time from (42) to show that:

Box 8. The River Model

FIGURE 9 In the river model of a black hole, fish that swim at different rates encounter a waterfall. The fastest fish represents a photon. "The fish upstream can make way against the current, but the fish downstream is swept to the bottom of the waterfall." The event horizon corresponds to that point on the waterfall at which the upward-swimming photon-fish stands still. [From Hamilton and Lisle, see references]

Andrew Hamilton and Jason Lisle created a **river model** of the black hole. In their model, water "looks like ordinary flat space, with the distinctive feature that space itself is flowing inward at the Newtonian escape velocity. The place where the infall velocity hits the speed of light . . . marks the event horizon. . . . Inside the event horizon, the infall velocity exceeds the speed of light, carrying everything with it." At every r-coordinate near a black hole the river of space flows past at the speed of a raindrop, namely a stone that falls from initial rest far from the waterfall.

Envision flat spacetime distant from a black hole as still water in a large lake with clocks that read raindrop wristwatch time τ_{raindrop} floating at rest with respect to the water. At one side of the lake the water drifts gently into a river and that carries the raindrop clocks with it. River water moves faster and faster as it approaches and flows over the brink of the waterfall. Each jet of falling water narrows as it accelerates downward. Fish represent objects that move in the river/space; the fastest fish represents a photon. At some point below the lip of the waterfall, not even the photon-fish can keep up with the downward flow and is swept to the bottom of the falls (Figure 9) The black hole event horizon corresponds to the point at which the upward-swimming "photon-fish" stands still.

The river model helps us to visualize many effects observed near the black hole. Hamilton and Lisle write, "It explains why light cannot escape from inside the event horizon, and why no star can come to rest within the event horizon. It explains how an extended object will be stretched radially by the inward acceleration of the river, and compressed transversely by the spherical convergence of the flow. It explains why an object that falls through the event horizon appears to an outsider redshifted and frozen at the event horizon: as the object approaches the event horizon, light [a photon-fish] emitted by it takes an ever-longer global time to forge against the onrushing current of space and eventually to reach the outside observer."

Hamilton and Lisle show that the river model is consistent with the results of general relativity. In that sense the river model is *correct* and *complete*.

The river model is a helpful visualization, but that visualization comes at a price. It carries two misleading messages: First, that space itself—represented by the river—is observable. We easily observe various flows of different rivers on Earth, but no one—and no instrument—registers or observes any "flow of space" into a black hole. Second, the river model embodies global rain coordinates, but we have seen that there are an unlimited number of global coordinates for the black hole, many of which cannot be envisioned by the river model.

$$\boxed{\frac{E_{\text{rain}}}{m} \equiv \lim_{\Delta\tau\to 0}\frac{\Delta t_{\text{rain}}}{\Delta\tau} \equiv \frac{dT}{d\tau} = \left(1-\frac{2M}{r}\right)^{-1}\left[\frac{E}{m}+\left(\frac{2M}{r}\right)^{1/2}\frac{dr}{d\tau}\right] \qquad (68)}$$

B. Is it possible for E_{rain}/m to become negative inside the event horizon? Would any observer complain if it did?

C. Same questions as Item B for E/m, the global map energy per unit mass.

D. Perform a Lorentz transformation (Section 1.10) with v_{rel} and γ_{rel} from (14) to obtain E_{shell} in terms of E. Compare with E and E_{shell} in Schwarzschild coordinates from Sections 6.2 and 6.3.

Box 9. The Planck length

General relativity is a classical—non-quantum—theory (Box 7, Section 6.7). One of its beauties is that, when applied to the black hole, general relativity points to its own limits. The Schwarzschild metric plus the Principle of Maximal Aging predict that everything which moves inward across the horizon will end up on the singularity, a point. We know that this does not satisfy quantum mechanics: The Heisenberg uncertainty principle of quantum mechanics tells us that a single electron confined to a point has unlimited momentum. So not even a single electron—much less an entire star gobbled up by the black hole—can be confined to the singularity. In this book we assume that classical general relativity is valid until very close to the singularity. How close? One estimate is the so-called **Planck length**, derived from three fundamental constants:

$$\text{Planck length} = \left(\frac{hG}{2\pi c^3}\right)^{1/2} = 1.616\,199 \times 10^{-35} \tag{69}$$

in meters, with an uncertainty of ± 97 in the last two digits. The presence in this equation of Planck's constant $h = 1.054\,571\,726 \times 10^{-34}$ Joule-second (± 15 in the last two digits) tells us that we have entered the realm of quantum mechanics, where classical general relativity is no longer valid. Cheer up! Before any part of you arrives at the Planck distance from the singularity, you will no longer feel any discomfort.

What happens when a single electron arrives at a Planck length away from the singularity? Nobody knows!

7.9 ■ A MERCIFUL ENDING?

How long does the "terminal spaghettification" process last?

Is death near the singularity painful?

To dive into a black hole is to commit suicide, which may go against religious, moral, or ethical principles—or against our survival instinct. Aside from such considerations, no one will volunteer for your black-hole diver research team if she predicts that as she approaches the crunch point her death will be painful. Your task is to estimate the **ouch time** τ_{ouch}, defined as the lapse of time on the wristwatch of the diver between her first discomfort and her arrival at the singularity, $r = 0$.

QUERY 19. Preliminary: Acceleration g in units of inverse meters. Newton's expression for gravitational force in conventional units:

$$F_{\text{conv}} \equiv m_{\text{conv}} g_{\text{conv}} = -\frac{GM_{\text{conv}} m_{\text{conv}}}{r^2} \qquad \text{(Newton)} \tag{70}$$

A. Verify the resulting gravitational acceleration in units of inverse meters:

$$g \equiv \frac{g_{\text{conv}}}{c^2} = -\frac{M}{r^2} \qquad \text{(Newton)} \tag{71}$$

B. Show that at Earth's surface the Newtonian acceleration of gravity has the value given inside the front cover, namely

$$|g_{\text{Earth}}| \equiv |g_{\text{E}}| = \left|-\frac{M_{\text{Earth}}}{r^2_{\text{Earth}}}\right| = 1.09 \times 10^{-16}\ \text{meter}^{-1} \qquad \text{(Newton)} \tag{72}$$

Rain frame observer feels tides.

The rain observer is in free fall and does not feel any net force as a result of local acceleration. However, she does feel radially stretched due to a *difference* in acceleration between her head and her feet, along with a compression from side to side. We call these differences **tidal accelerations**.

QUERY 20. Tidal acceleration along the r-coordinate line

We want to know how much this acceleration *differs* between the head and the feet of an in-falling rain observer. Take the differential of g in (71). Convert the result to increments over a patch of average $\bar{r}$. Show that

$$\Delta g \approx \frac{2M}{\bar{r}^3}\Delta r \qquad \text{(Newton)} \tag{73}$$

What does Einstein say about tidal acceleration? Section 9.7 displays the correct general relativistic expressions for the variation of local gravity with spatial separation. *Surprise:* The expression for tidal acceleration in *any* inertial frame falling along the r-coordinate line has a form identical to the Newtonian result (73), and thus for the local rain frame becomes:

$$\Delta g_{\text{rain}} \approx \frac{2M}{\bar{r}^3}\Delta y_{\text{rain}} \tag{74}$$

Define "discomfort."

What are the criteria for discomfort? Individual rain observers will have different tolerance to tidal forces. To get a rough idea, let the rain observer's body be oriented along the r-coordinate line as she falls, and assume that her stomach is in free fall, feeling no stress whatever. Assume that the rain observer first becomes uncomfortable when the *difference* in local acceleration between her free-fall stomach and her head (or her feet), that stretches her, is equal to the acceleration at Earth's surface. By this definition, the rain observer becomes uncomfortable when her feet are pulled downward with a force equal to their weight on Earth and her head is pulled upward with a force of similar magnitude. Let her height be h in her frame, and the ruler distance between stomach and either her head or her feet be half of this, that is, $\Delta y_{\text{rain}} = h/2$.

QUERY 21. The r-value for the start of "ouch."

From our criteria above for discomfort, we have:

$$\Delta g_{\text{rain ouch}} \equiv g_{\text{E}} \qquad \text{(stomach-to-foot distance)} \tag{75}$$

Show that the r-value for the start of "ouch," namely r_{ouch}, is:

$$r_{\text{ouch}} = \left(\frac{Mh}{g_{\text{E}}}\right)^{1/3} \qquad (h = \text{head-to-foot height}) \tag{76}$$

QUERY 22. Three cases for the start of "ouch."
Approximate h, the head-to-foot distance, as 2 meters. Find the value of the ratio $r_{\text{ouch}}/(2M)$ for these cases:

A. A black hole with ten times the mass of our Sun.
B. The "20-year black hole" in Query 3.
C. Suppose that the ouch r-coordinate is at the event horizon. What is the mass of the black hole as a multiple of the Sun's mass?

QUERY 23. The wristwatch ouch time τ_{ouch}

A. Use equation (2) to show that the raindrop ouch time τ_{ouch} (the wristwatch time between initial ouch and arrival at the singularity) is *independent of the mass of the black hole*:

$$\tau_{\text{ouch}} = \frac{1}{3}\left(\frac{2h}{g_{\text{E}}}\right)^{1/2} \qquad \text{(raindrop wristwatch ouch time in meters)} \qquad (77)$$

Here, recall, h is the height of the astronaut, about 2 meters, and the value of g_{E} is given in (72). Show that the wristwatch ouch time in seconds, the same for *all* non-spinning black holes, is:

$$\tau_{\text{ouch}} = \frac{1}{3c}\left(\frac{2h}{g_{\text{E}}}\right)^{1/2} \qquad \text{(raindrop wristwatch ouch time in seconds)} \qquad (78)$$

B. Substitute numbers into equation (78). Show that the duration of raindrop wristwatch ouch time is about 2/9 of a second for *every* non-spinning black hole, independent of its mass M. *Guess:* Will pain signals travel from your extremities to your brain during this brief wristwatch ouch time?

MUTABILITY OF PHYSICAL LAWS

By 1970, I had become convinced not only that black holes are an inevitable consequence of general relativity theory and that they are likely to exist in profusion in the universe, but also that their existence implies the mutability of physical law. If time can end in a black hole, if space can be crumpled to nothingness at its center, if the number of particles within a black hole has no meaning, then why should we believe that there is anything special, anything unique, about the laws of physics that we discover and apply? These laws must have come into existence with the Big Bang as surely as space and time did.

—John Archibald Wheeler

7.10 ■ EXERCISES

1. Crossing the Event Horizon

Pete Brown disagrees with the statement, "No special event occurs as we fall through the event horizon." He says, "Suppose you go feet first through the event horizon. Since your feet hit the event horizon before your eyes, then your feet should disappear for a short time on your wristwatch. When your eyes pass across the event horizon, you can see again what's inside, including your feet. So tie your sneakers tightly or you will lose them in the dark!" Is Pete correct? Analyze his argument without criticizing him.

2. Equations of Motion of the Raindrop

From equations in Chapter 6 we can derive the equations of motion for a raindrop in Schwarzschild coordinates. From the definition of the raindrop,

$$\frac{E}{m} = 1 \quad \text{and} \quad \frac{d\phi}{d\tau} = 0 \qquad \begin{array}{c}\text{(raindrop in Schwarzschild coordinates} \\ \text{and in global rain coordinates)}\end{array} \tag{79}$$

In addition, equation (23) in Section 6.4 tells us that

$$\frac{dr}{d\tau} = -\left(\frac{2M}{r}\right)^{1/2} \qquad \begin{array}{c}\text{(raindrop in Schwarzschild coordinates} \\ \text{and in global rain coordinates)}\end{array} \tag{80}$$

From equation (13) in Section 6.4, you can easily show that

$$\frac{dt}{d\tau} = \left(1 - \frac{2M}{r}\right)^{-1} \qquad \begin{array}{c}\text{(raindrop in Schwarzschild coordinates} \\ \text{and in global rain coordinates)}\end{array} \tag{81}$$

Now derive the raindrop equations of motion in global rain coordinates. First, show that both $dr/d\tau$ and $d\phi/d\tau$ have the same form in global rain coordinates as in Schwarzschild coordinates, as stated in the labels of equations (80) and (81). Second, use equations (80) and (81) plus equation (35) to show that

$$\frac{dT}{d\tau} = 1 \qquad \text{(raindrop in global rain coordinates)} \tag{82}$$

Comment 9. Simple definition of global rain T
Equation (82) can be used as the definition of global rain coordinate differential dT. In other words, we *choose* dT equal to the differential lapse of wristwatch time on a falling raindrop.

3. Different masses for the "20-year black hole."

This chapter describes a "20-year black hole," defined as one for which the wristwatch on a raindrop registers a 20-year lapse between its crossing of the event horizon and its arrival at the singularity. But the wristwatch may be on a hailstone, flung radially inward from far away; or on a drip, dropped from rest from a shell outside the event horizon. What is the required mass of the "20-year black hole" in these two cases?

A. We fling an incoming **hailstone** inward along the r-coordinate line with initial shell speed $|v_{\rm far}|$ from far away from the black hole. A lengthy derivation of the wristwatch time from event horizon to the singularity yields the result:

$$\tau_{\rm Aail}[2M \rightarrow 0] = M\left[\frac{2}{v_{\rm far}^2\gamma_{\rm far}} - \frac{1}{v_{\rm far}^3\gamma_{\rm far}^3}\ln\left(\frac{1+v_{\rm far}}{1-v_{\rm far}}\right)\right] \qquad (83)$$

where, remember, $\gamma \equiv (1-v^2)^{-1/2}$ and we treat $v_{\rm far}$ as a (positive) speed. Answer questions in the following items:

a. *Guess:* In the case of the hailstone, will the mass of "20-year black hole" be greater or less than that for the raindrop?

b. Consider $\gamma_{\rm far} = 2$. What is the value of $v_{\rm far}$?

c. Show that for this particular value $\gamma_{\rm far} = 2$, the first term inside the square bracket in (83) alone gives the same result as the raindrop in equation (3).

d. Was your guess in Item **a** correct or incorrect?

e. What is the mass of the "20-year black hole" for that hailstone? How does it compare to the mass of the "20-year black hole" for the raindrop?

B. A **drip** drops from rest on a shell of global coordinate $r_0 > 2M$. Another lengthy derivation of the wristwatch time from event horizon to the singularity yields the result:

$$\tau_{\rm drip}[2M \rightarrow 0] = \qquad (84)$$

$$2M\left(\frac{2M}{r_0}\right)^{-3/2}\left[-\left(\frac{2M}{r_0}\right)^{1/2}\left(1-\frac{2M}{r_0}\right)^{1/2} + \arctan\left(\frac{2M/r_0}{1-2M/r_0}\right)^{1/2}\right]$$

a. *Guess:* In the case of a drip, will the mass of "20-year black hole" be greater or less than that for the raindrop?

b. Next, take the limiting case $r_0 \rightarrow 2M$. Show that in this limit arctan takes the value $\pi/2$.

c. Show that in this case $\tau_{\rm drip}[2M \rightarrow 0] \rightarrow \pi M$.

d. Was your guess in Item **a** correct or incorrect?

e. What is the mass of the "20-year black hole" for that drip? How does it compare to the mass of the "20-year black hole" for the raindrop?

C. *Fascinating but optional:* The next to last paragraph in Box 8 states that every stone that passes inward across the event horizon at $r = 2M$ moves at that r-value with shell velocity $v_{\text{shell}} = -1$, the speed of light (as a limiting case). Since this holds for *all* r-diving stones, how can the masses of "20-year black holes" possibly differ for raindrops, hailstones, and drips?

4. Map energy of a drip released from r_0

A. Derive the following expression for E/m in global rain coordinates for a drip released from rest with respect to the local shell frame at $r_0 > 2M$:

$$\frac{E}{m} = \left(1 - \frac{2M}{r_0}\right)^{1/2} \qquad \text{(drip released from rest at } r_0) \qquad (85)$$

Compare with equation (33) of Chapter 6. Are you surprised by what you find? Should you be?

B. What are the maximum and minimum values of E/m in (85) as a function of r_0? How can the minimum value possibly be less than the rest energy m of the stone measured in an inertial frame?

C. Is expression (85) consistent with the value $E/m = 1$ for a raindrop?

D. Is expression (85) valid for $r_0 < 2M$? What is the physical reason for your answer?

E. For $r_0 > 2M$, is expression (85) still valid when that stone arrives inside the event horizon?

5. Map energy of a hailstone

A. Derive the following expression for E/m in global rain coordinates for a hailstone hurled radially inward with speed v_{far} from a shell very far from the black hole.

$$\frac{E}{m} = \gamma_{\text{far}} \equiv \left(1 - v_{\text{far}}^2\right)^{-1/2} \qquad \text{(hailstone)} \qquad (86)$$

Compare this expression with results of Exercise 7 in Chapter 6. Are you surprised by what you find? Should you be?

B. Is expression (86) for the hailstone consistent with expression (85) for the drip? consistent with $E/m = 1$ for a raindrop?

6. Motion of outgoing light flash outside and at the event horizon

Find the maximum value of r/M at which the "outgoing" flash moves to larger r, that is $dr > 0$, at each of these global map velocities:

A. $dr/dT = 0.99$
B. $dr/dT = 0.9$
C. $dr/dT = 0.5$
D. $dr/dT = 0$

7. Motion of the "outgoing" flash inside the event horizon.

Find the value of r/M at which the"outgoing" flash moves to smaller r, that is $dr < 0$, at each of these global map velocities:

A. $dr/dT = -0.1$
B. $dr/dT = -0.5$
C. $dr/dT = -1$
D. $dr/dT = -9$

8. Motion of the incoming flash

At each value of r/M found in Exercises 6 and 7, find the value of dr/dT for the *incoming* flash.

7.11 ■ REFERENCES

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