

Chapter 5. Global and Local Metrics

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- 14 • *Why did Einstein take seven years to go from special relativity to general*
- 15 *relativity?*
- 16 • *Why are so many different kinds of flat maps used to plot Earth's curved*
- 17 *surface?*
- 18 • *Why use coordinates at all? Why not just measure distances directly, say*
- 19 *with a ruler?*
- 20 • *Why does the spacetime metric use differentials?*
- 21 • *Are Schwarzschild global coordinates the only way to describe spacetime*
- 22 *around a black hole?*

CHAPTER

5

Global and Local Metrics

Edmund Bertschinger & Edwin F. Taylor *

The basic demand of the special theory of relativity (invariance of the laws under Lorentz-transformations) is too narrow, i.e., that an invariance of the laws must be postulated relative to nonlinear transformations for the co-ordinates in the four-dimensional continuum.

*This happened in 1908. Why were another seven years required for the construction of the general theory of relativity? **The main reason lies in the fact that it is not so easy to free oneself from the idea that coordinates must have an immediate metrical meaning.***

—Albert Einstein [boldface added]

5.1 ■ EINSTEIN'S PERPLEXITY

Why seven years between special relativity and general relativity?

Einstein's seven-year puzzle

It took Albert Einstein seven years to solve the puzzle compressed into the two-paragraph quotation above. The first paragraph complains that special relativity (with its restriction to flat spacetime coordinates) is too narrow. Einstein demands that a *nonlinear* coordinate system—that is, one that is *arbitrarily stretched*—should also be legal. *Nonlinear* means that it can be stretched by different amounts in different locations.

Stretch coordinates arbitrarily.

In the second paragraph, Einstein explains his seven-year problem: He tried to apply to a stretched coordinate system the same rules used in special relativity. Einstein's phrase **immediate metrical meaning** describes something that can be measured directly—for example, the radar-measured distance between the top of the Eiffel Tower and the Paris Opera building. Einstein says that since we can use nonlinear stretched coordinates, these coordinate separations need not be something we can measure directly, for example with a ruler.

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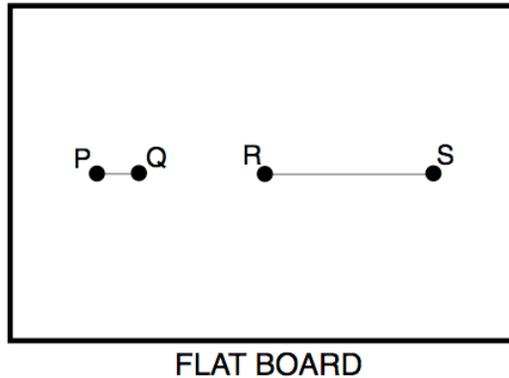


FIGURE 1 Compare distances between two different pairs of points on a flat wooden cutting board. First measure with a ruler the distance between the pair of points P and Q. Then measure the distance between the pair of points R and S. Measured distance PQ is *smaller* than the measured distance RS. We require no coordinate system whatsoever to verify this inequality; we measure distances directly on a flat surface.

Solving Einstein's puzzle leads to the global metric.

52 What is the relation between the coordinate separations between two
 53 points and the directly-measured distance between those two points? How
 54 does this distinction affect predictions of special and general relativity?
 55 Answering these questions reveals the unmeasurable nature of global
 56 coordinate separations, but nevertheless the central role of the *global metric* in
 57 connecting different local inertial frames in which we carry out measurements.

5.2.2 ■ EINSTEIN'S PERPLEXITY ON A WOODEN CUTTING BOARD

59 *Move beyond high school geometry and trigonometry!*

Simplify: From curved spacetime to a flat cutting board.

60 We transfer Einstein's puzzle from spacetime to space and—to simplify
 61 further—measure the distance between two points on the flat surface of a
 62 wooden cutting board (Figure 1).

Measure distance directly, with a ruler.

63 A pair of points, P and Q, lie near to one another on the surface. A second
 64 pair of points, R and S, are farther apart than points P and Q. How do we
 65 know that distance RS is greater than distance PQ? We measure the two
 66 distances directly, with a ruler. To ensure accuracy, we borrow a ruler from the
 67 local branch of the National Institute of Standards and Technology. Sure
 68 enough, with our official centimeter-scale ruler we verify distance RS to be
 69 greater than distance PQ. *We do not need any coordinate system whatsoever*
 70 *to measure distance PQ or distance RS or to compare these distances on a flat*
 71 *surface.*

Difference in Cartesian coordinates verifies difference in distances.

72 Next, apply coordinates to the flat surface. Do not draw coordinate lines
 73 directly on the cutting board; instead spread a fishnet over it (Figure 2). When
 74 we first lay down the fishnet, its narrow strings look like Cartesian square
 75 coordinate lines. Adjacent strings are one centimeter apart. The *x*-coordinate
 76 separation between P and Q is 1 centimeter, and the *x*-coordinate separation

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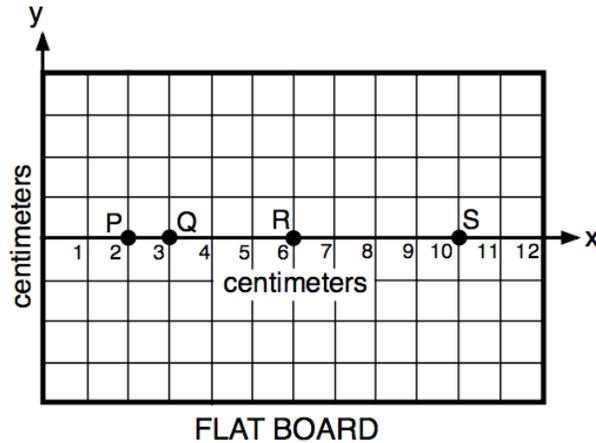


FIGURE 2 A fishnet with one-centimeter separations covers the wooden cutting board. Expressed in these coordinates, the coordinate separation PQ is 1 centimeter, while the coordinate separation RS is 4 centimeters. In this case a coordinate separation *does* have “an immediate metrical meaning” in Einstein’s phrase. *Interpretation:* In this case we can derive from coordinate separations the values of directly-measured distances.

Stretch fishnet by variable amounts in x -direction.

“Stretch” coordinate separation not equal to measured distance.

Stretch coordinates form a legal map.

77 between R and S is 4 centimeters, confirming the inequality in our direct
 78 distance measurements. In this case each difference (or separation) in
 79 Cartesian coordinates, PQ and RS, *does* have “an immediate metrical
 80 meaning;” in other words, it corresponds to the *directly-measured distance*.

81 Moving ahead, suppose that instead of string, we make the fishnet out of
 82 rubber bands. As we lay the rubber band fishnet loosely on the cutting board,
 83 we do something apparently screwy: As we tack down the fishnet, we stretch it
 84 along the x -direction by different amounts at different horizontal positions.
 85 Figure 3 shows the resulting “stretch” coordinates along the x -direction.

86 Now check the x -coordinate difference between P and Q in Figure 3, a
 87 difference that we call Δx_{PQ} . Then $\Delta x_{PQ} = 5 - 2 = 3$. Compare this with the
 88 x -coordinate separation between R and S: $\Delta x_{RS} = 10 - 9 = 1$. Lo and behold,
 89 the coordinate separation Δx_{PQ} is *greater* than the coordinate separation
 90 Δx_{RS} , even though our directly-measured distance PQ is *less* than the
 91 distance RS. This contradiction is the simplest example we can find of the
 92 great truth that Einstein grasped after seven years of struggle: *coordinate*
 93 *separations need not be directly measurable*.

94 “No fair!” you shout. “You can’t just move coordinate lines around
 95 arbitrarily like that.” Oh yes we can. Who is to prevent us? Any coordinate
 96 system constitutes a **map**. What is a map? Applied to our cutting board, a
 97 map is simply a rule for assigning numbers that uniquely specify the location
 98 of every individual point on the surface. Our coordinate system in Figure 3
 99 does that job nicely; it is a legal and legitimate map. However, the amount of
 100 stretching—what we call the **map scale**—varies along the x -direction.

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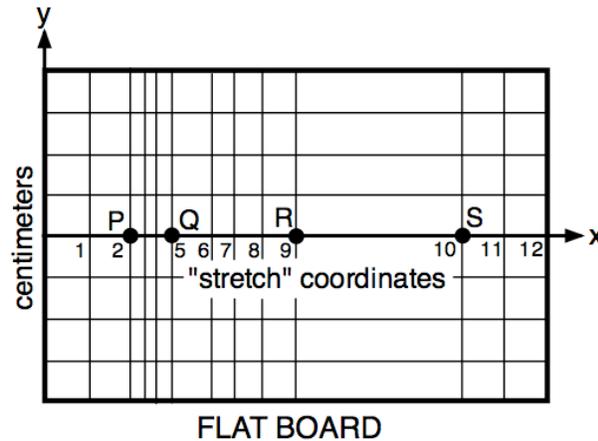


FIGURE 3 Global coordinate system that covers our entire cutting board, but in this case made with a rubber fishnet tacked down so as to stretch the x separation of fishnet cords by different amounts at different locations along the horizontal direction. The coordinate separation $\Delta x_{PQ} = 3$ between points P and Q is greater than the coordinate separation $\Delta x_{RS} = 1$ between points R and S, even though the measured *distances* between each of these pairs show the reverse inequality. Einstein was right: In this case coordinate separations do *not* have “an immediate metrical meaning;” in other words, coordinate separations do *not* tell us the values of directly-measured distances.

101 Of course, for convenience we usually *choose* the map scale to be
 102 everywhere uniform, as displayed in Figure 2. This choice is perfectly legal. We
 103 call this legality of Cartesian coordinates Assertion 1:

**Assertion 1 for a
 FLAT SURFACE:
 CAN draw map with
 everywhere-uniform
 map scale.**

104 **Assertion 1. ON A FLAT SURFACE IN SPACE, we CAN FIND a global**
 105 **coordinate system such that every coordinate separation IS a**
 106 **directly-measured distance.**

107 Standard Cartesian (x, y) coordinates allow us to use the power of the
 108 Pythagorean Theorem to predict the directly-measured distance s between two
 109 points anywhere on the board in Figure 2:

$$\Delta s^2 = \Delta x^2 + \Delta y^2 \quad (\text{flat surface: Choose Cartesian coordinates.}) \quad (1)$$

Cartesian separations:
 Pythagoras works!

110 The coordinate separations Δx and Δy and the resulting measured distance
 111 Δs can be as small or as large as we want, as long as the map scale is uniform
 112 everywhere on the flat cutting board.

113 In contrast, we *cannot* apply the Pythagorean Theorem using the
 114 “stretch” coordinates in Figure 3 to find the distance between a pair of points
 115 that are far apart in the x -direction. Why not? Because a large separation
 116 between two points can span regions where the map scale varies noticeably,
 117 that is, where rubber bands stretch by substantially different amounts. For
 118 example in Figure 3, the x -coordinate separation between points Q and S on

Section 5.3 Global space metric for a flat surface **5-5**

Stretch coordinates:
Pythagoras fails
on a flat surface.

119 the flat surface is $\Delta x_{QS} = 5$, whereas points P and S have a much greater
120 x -coordinate separation: $\Delta x_{PS} = 8$. This is true even though the
121 directly-measured *distance* between P and S is only slightly greater than the
122 directly-measured *distance* between Q and S.

123 Stretched-fishnet coordinates of Figure 3, provide a case in which the
124 Pythagorean Theorem (1) gives incorrect answers—coordinate separations are
125 *not* the same as directly-measured distances. This yields Assertion 2, an
126 alternative to Assertion 1:

**Assertion 2 for a
FLAT SURFACE:
We are FREE to
choose variable
map scale over
the surface.**

127 **Assertion 2. ON A FLAT SURFACE IN SPACE, we are FREE TO CHOOSE a**
128 **global coordinate system for which coordinate separations ARE NOT**
129 **directly-measured distances.**

5.3. GLOBAL SPACE METRIC FOR A FLAT SURFACE

131 *Space metric to the rescue.*

How can we predict
measured distances
using arbitrary
coordinates?
Answer: The metric!

132 Einstein tells us that we are free to stretch or contract conventional (in this
133 case Cartesian) coordinates in any way we want. But if we do, then the
134 resulting coordinate separations lose their “immediate metrical meaning;” that
135 is, a coordinate separation between a pair of points no longer predicts the
136 distance we measure between these points. If the coordinate separation can no
137 longer tell us the distance between two points, what can? Our simple question
138 about space on a flat cutting board is a preview of the far more profound
139 question about spacetime with which Einstein struggled: How can we predict
140 the measured wristwatch time τ or the measured ruler distance σ between a
141 pair of events using the differences in *arbitrary* global coordinates between
142 them? The answer was a breakthrough: “The metric!” Here’s the path to that
143 answer, starting with our little cutting board.

Space metric
gives differential ds
from differentials
 dx and dy .

144 Begin by recognizing that very close to any point on the flat surface the
145 coordinate scale is nearly uniform, with a multiplying factor (local map scale)
146 to correct for the local stretching in the x -coordinate. Strictly speaking, the
147 coordinate scale is uniform only vanishingly close to a given point. *Vanishingly*
148 *close?* That phrase instructs us to use the vanishingly small calculus limit:
149 differential coordinate separations. For the coordinates of Figure 3, we find the
150 differential distance ds from a **global space metric** of the form:

$$ds^2 = F(x_{\text{stretch}})dx_{\text{stretch}}^2 + dy_{\text{stretch}}^2 \quad (\text{variable } x\text{-stretch}) \quad (2)$$

151 To repeat, we use the word *global* to emphasize that x is a valid coordinate
152 everywhere across our cutting board covered by the stretched fishnet. In (2),
153 $F(x)$ —actually the square root of $F(x)$ —is the map scale that corrects for the
154 stretch in the horizontal coordinate *differentially close to that value of x* . If
155 $F(x)$ is defined everywhere on the cutting board, however, then equation (2) is
156 also valid at every point on the board.

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Metric works well
LOCALLY, even
with stretched
coordinates.

157 The global space metric is a tremendous achievement. On the right side of
158 metric (2) the function $F(x)$ corrects the squared differential dx_{stretch}^2 to give
159 the correct squared differential distance ds^2 on the left side.

Differential distance
 ds is too small
to measure. . .

160 We have gained a solution to Einstein’s puzzle for the simplified case of
161 differential separations on a flat surface in space. But we seem to have suffered
162 a great loss as well: calculus insists that the differential distance ds predicted
163 by the space metric is vanishingly small. We cannot use our official
164 centimeter-scale ruler to measure a vanishingly small differential distance. How
165 can we possibly predict a measured distance—for example the distance
166 between points P and S on our flat cutting board? We want to predict and
167 then make *real* measurements on *real* flat surfaces!

. . . but we can predict
measured distance
from summed
(integrated) ds .

168 Differential calculus curses us with its stingy differential separations ds ,
169 but integral calculus rescues us. We can sum (“integrate”) differential
170 distances ds along the curve. The result is a predicted *total distance* along the
171 curved path, a prediction that we can verify with a tape measure. As a special
172 case, let’s predict the distance s along the straight horizontal x -axis from point
173 P to point S in Figure 3. Call this distance s_{PS} . “Horizontal” means no
174 vertical, so that $dy = 0$ in equation (2). The distance s_{PS} is then the sum
175 (integral) of $ds = [F(x)]^{1/2} dx$ from $x = 2$ to $x = 10$, where the scale function
176 $[F(x)]^{1/2}$ varies with the value of x :

$$s_{\text{PS}} = \int_{x=2}^{x=10} [F(x_{\text{stretch}})]^{1/2} dx_{\text{stretch}} \quad (\text{horizontal distance: P to S}) \quad (3)$$

177 When we evaluate this integral, we can once again use our official
178 centimeter-scale ruler to verify by direct measurement that the total distance
179 s_{PS} between points P and S predicted by (3) is correct.

180 The example of metric (2) leads to our third important assertion:

**Assertion 3 for a
FLAT SURFACE:**
Metric gives us ds ,
whose integral predicts
measured distance s .

181 **Assertion 3. ON A FLAT SURFACE IN SPACE when using a global**
182 **coordinate system for which coordinate separations ARE NOT**
183 **directly-measured distances, a space metric is REQUIRED to give the**
184 **differential distance ds whose integrated value predicts the measured**
185 **distance s between points.**

5.4 ■ GLOBAL SPACE METRIC FOR A CURVED SURFACE

187 *Squash a spherical map of Earth’s surface onto a flat table? Good luck!*

188 In Sections 5.2 and 5.3, we chose variably-stretched coordinates on a flat
189 surface. Then we corrected the effects of the variable stretching using a metric.
190 This is a cute mathematical trick, but who cares? We are not *forced* to use
191 stretched coordinates on a flat cutting board, so why bother with them at all?
192 To answer these questions, apply our ideas about maps to the curved surface
193 of Earth. Chapter 2 derived a global metric—equation (3), Section 2.3—for
194 the spherical surface of Earth using angular coordinates λ for latitude and ϕ

Section 5.4 global space metric for a curved surface **5-7**

195 for longitude, along with Earth’s radius R . Here we convert that global metric
196 to coordinates x and y :

$$\begin{aligned}
 ds^2 &= R^2 \cos^2 \lambda d\phi^2 + R^2 d\lambda^2 && (0 \leq \phi < 2\pi \text{ and } -\pi/2 \leq \lambda \leq \pi/2) && (4) \\
 &= \cos^2 \left(\frac{R\lambda}{R} \right) (Rd\phi)^2 + (Rd\lambda)^2 && \text{(metric : Earth's surface)} \\
 &= \cos^2 \left(\frac{y}{R} \right) dx^2 + dy^2 && (0 \leq x < 2\pi R \text{ and } -\pi R/2 \leq y \leq \pi R/2)
 \end{aligned}$$

197 On a sphere, we define $y \equiv R\lambda$ and $x \equiv R\phi$ (the latter from the definition of
198 radian measure).

Undistorted flat maps of Earth impossible.

199 Compare the third line of (4) with equation (2). The y -dependent
200 coefficient of dx^2 results from the fact that as you move north or south from
201 the equator, lines of longitude converge toward a single point at each pole.
202 That coefficient of dx^2 makes it impossible to cover Earth’s spherical surface
203 with a flat Cartesian map without stretching or compressing the map at some
204 locations.

A curved surface forces us to use stretched coordinates.

205 Throughout history, mapmakers have struggled to create a variety of flat
206 projections of Earth’s spherical surface for one purpose or another. But each
207 projection has *some* distortion. *No uniform projection of Earth’s surface can*
208 *be laid on a flat surface without stretching or compression in some locations.* If
209 this is impossible for a spherical Earth with its single radius of curvature, it is
210 certainly impossible for a general curved surface—such as a potato—with
211 different radii of curvature in different locations. In brief, it is impossible to
212 completely cover a curved surface with a single Cartesian coordinate system.
213 (Is a cylindrical surface curved? No; technically it is a flat surface, like a
214 rolled-up newspaper, which Cartesian coordinates can map exactly.) We
215 bypass formal proof and state the conclusion:

Assertion 4 for a CURVED SURFACE: Everywhere-uniform map scale is IMPOSSIBLE.

216 **Assertion 4. ON A CURVED SURFACE IN SPACE, it is IMPOSSIBLE to find a**
217 **global coordinate system for which coordinate separations EVERYWHERE**
218 **on the surface are directly-measured distances.**

Metric required on curved surface.

219 The dy on the third line of equation (4) is still a directly-measured
220 distance: the differential distance northward from the equator. That is true for
221 a sphere, whose constant R -value allows us to define $y \equiv R\lambda$. But Earth is not
222 a perfect sphere; rotation on its axis results in a slightly-bulging equator.
223 Technically the Earth is an **oblate spheroid**, like a squashed balloon. In that
224 case neither x or y coordinate separations are directly-measured distances.
225 And most curved surfaces are more complex than the squashed balloon.
226 Einstein was right: In most cases coordinate separations *cannot* be
227 directly-measurable distances.

228 No possible uniform map scale over the entire surface of Earth? Then
229 there is an inevitable distinction between a coordinate separation and
230 measured distance. The space metric is no longer just an option, but has
231 become the indispensable practical tool for predicting distances between two
232 points from their coordinate separations.

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Assertion 5 for a
CURVED SURFACE:
Metric REQUIRED
to calculate distance.

233 **Assertion 5. ON A CURVED SURFACE IN SPACE, a global space metric is**
234 **REQUIRED to calculate the differential distance ds between a pair of**
235 **adjacent points from their differential coordinate separations.**

236 As before, integrating the differential ds yields a measured total distance s
237 along a path on the curved surface, whose predicted length we can verify
238 directly with a tape measure.

Space
summary

239 **SPACE SUMMARY:** *On a flat surface in space we can choose*
240 *Cartesian coordinates, so that the Pythagorean theorem—with no*
241 *differentials—correctly predicts the distance s between two points*
242 *far from one another. On a curved surface we cannot. But on any*
243 *curved surface we can use a space metric to calculate ds between a*
244 *pair of adjacent points from values of the differential coordinate*
245 *separations between them. Then we can integrate these differentials*
246 *ds along a given path in space to predict the directly-measured*
247 *length s along that path.*

“Connectedness”
= topology.

248 The combination of global coordinates plus the global metric is even more
249 powerful than our summary implies. Taken together, the two describe a curved
250 surface completely. In principle we can use the global coordinates plus the
251 metric to reconstruct the curved surface exactly. (Strictly speaking, the global
252 coordinate system must include information about ranges of its coordinates,
253 ranges that describe its “connectedness”—technical name: its **topology**.)

5.5 ■ GLOBAL SPACETIME METRIC

255 *Visit a neutron star with wristwatch, tape measure—and metric—in your back pocket.*
256

To distorted space
add warped t .
Result? Trouble
for Einstein!

257 What does all this curved-surface-in-space talk have to do with Einstein’s
258 perplexity during his journey from special relativity to general relativity? As
259 usual, we express the answer as an analogy between a curved surface in space
260 and a curved region of spacetime. Spacetime around a black hole multiplies the
261 complications of the curved surface: not only is space distorted compared with
262 its Euclidean description but the fourth dimension, the t -coordinate, is warped
263 as well. All this complicates our new task, which is to predict our measurement
264 of ruler distance σ or wristwatch time τ between a *pair of events in spacetime*.

265 Here we simply state, for flat and curved regions of spacetime, five
266 assertions similar to those stated earlier for flat and curved surfaces in space.

Assertion A for
FLAT SPACETIME:
Everywhere-uniform
map scale possible.

267 **Assertion A. IN A FLAT REGION OF SPACETIME, we CAN FIND a global**
268 **coordinate system in which every coordinate separation IS a**
269 **directly-measured quantity.**

270 In Chapter 1 we introduced a pair of expressions for flat spacetime called the
271 *interval*, similar to the Pythagorean Theorem for a flat surface. One form of
272 the interval predicts the wristwatch time τ between two events with a timelike

Section 5.5 global spacetime metric **5-9**

273 relation. The second form tells us the ruler distance σ between two events with
 274 a spacelike relation:

$$\Delta\tau_{\text{lab}}^2 = \Delta t_{\text{lab}}^2 - \Delta s_{\text{lab}}^2 \quad (\text{flat spacetime, timelike-related events}) \quad (5)$$

$$\Delta\sigma^2 = \Delta s_{\text{lab}}^2 - \Delta t_{\text{lab}}^2 \quad (\text{flat spacetime, spacelike-related events})$$

275 In *flat* spacetime, each space coordinate separation Δs_{lab} and time coordinate
 276 separation Δt_{lab} measured in the laboratory frame can be as small or as great
 277 as we want. On to our second assertion:

**Assertion B for
 FLAT SPACETIME:
 We are free to choose
 a variable map scale
 over the region.**

278 **Assertion B. IN A FLAT REGION OF SPACETIME we are FREE TO CHOOSE**
 279 **a global coordinate system in which coordinate separations**
 280 **ARE NOT directly-measured quantities.**

281 In this case we can choose not only stretched space coordinates but also a
 282 system of scattered clocks that run at different rates. If we choose such a
 283 “stretched” (but perfectly legal) global spacetime coordinate system, the
 284 interval equations (5) are no longer valid, because any of these coordinate
 285 separations may span regions of varying spacetime map scales. So we again
 286 retreat to a differential version of this equation, adding coefficients similar to
 287 that of space metric (2). A simple timelike metric might have the general form:

$$d\tau^2 = J(t, y, x)dt^2 - K(t, y, x)dy^2 - L(t, y, x)dx^2 \quad (6)$$

Spacetime metric
 delivers $d\tau$ from
 differentials dt ,
 dy , and dx .

288 Here each of the coefficient functions J , K , and L may vary with x , y , and t .
 289 (The coefficient functions are not entirely arbitrary: the condition of flatness
 290 imposes differential relations between them, which we do not state here.)
 291 Given such a metric for flat spacetime, we are free to use this metric to
 292 convert differentials of global coordinates (right side of the metric) to
 293 measured quantities (left side of the metric). This leads to our third assertion:

**Assertion C for
 FLAT SPACETIME:
 Variable map scale
 requires metric
 to calculate
 $d\tau$ or $d\sigma$.**

294 **Assertion C. IN A FLAT REGION OF SPACETIME, when we choose a global**
 295 **coordinate system in which coordinate separations are not**
 296 **directly-measured quantities, then a global spacetime metric is REQUIRED**
 297 **to calculate the differential interval, $d\tau$ or $d\sigma$, between two adjacent events**
 298 **using their differential global coordinate separations.**

299 On the other hand, in a region of curved spacetime—analogueous to the
 300 situation on a curved surface in space—we *cannot* set up a global coordinate
 301 system with the same map scale everywhere in the region.

**Assertion D for
 CURVED
 SPACETIME:
 Everywhere-uniform
 map scale is
 IMPOSSIBLE.**

302 **Assertion D. IN A CURVED REGION OF SPACETIME it is IMPOSSIBLE to**
 303 **find a global coordinate system in which coordinate separations**
 304 **EVERYWHERE in the region are directly-measured quantities.**

**Assertion E for
 CURVED
 SPACETIME:
 Metric REQUIRED
 to calculate
 $d\tau$ or $d\sigma$.**

305 **Assertion E. IN A CURVED REGION OF SPACETIME, a global spacetime**
 306 **metric is REQUIRED to calculate the differential interval, $d\tau$ or $d\sigma$, between**
 307 **a pair of adjacent events from their differential global coordinate**
 308 **separations.**

5-10 Chapter 5 Global and Local MetricsSpacetime
summary

309 **SPACETIME SUMMARY:** *In flat spacetime we can choose*
 310 *coordinates such that the spacetime interval—with no*
 311 *differentials—correctly predicts the wristwatch time (or the ruler*
 312 *distance) between two events far from one another. In curved*
 313 *spacetime we cannot. But in curved spacetime we can use a*
 314 *spacetime metric to calculate $d\tau$ or $d\sigma$ between adjacent events*
 315 *from the values of the differential coordinate separations between*
 316 *them. Then we can integrate $d\tau$ along the worldline of a particle,*
 317 *for example, to predict the directly-measured time lapse τ on a*
 318 *wristwatch that moves along that worldline.*

“Connectedness”
= topology.

319 As in the case of the curved surface, a complete description of a spacetime
 320 region results from the combination of global spacetime coordinates and global
 321 metric—along with the connectedness (topology) of that region. For example,
 322 we can in principle use Schwarzschild’s global coordinates and his metric to
 323 answer all questions about spacetime around the black hole.

5.6. ■ ARE WE SMARTER THAN EINSTEIN?325 *Did Einstein fumble his seven-year puzzle?*

Einstein’s struggle

326 We have now solved the puzzle that troubled Einstein for the seven years it
 327 took him to move from special relativity to general relativity. Surely Einstein
 328 would understand in a few seconds the central idea behind cutting-board
 329 examples in Figures 1 through 3. However, the extension of this idea to the
 330 four dimensions of spacetime was not obvious while he was struggling to create
 331 a brand new theory of spacetime that is curved, for example, by the presence
 332 of Earth, Sun, neutron star, or black hole. Is it any wonder that during this
 333 intense creative process Einstein took a while to appreciate the lack of
 334 “immediate metrical meaning” of differences in global coordinates?

One co-author
didn’t get it.

335 It is embarrassing to admit that one co-author of this book (EFT)
 336 required more than two years to wake up to the basic idea behind the present
 337 chapter, even though this central result is well known to every practitioner of
 338 general relativity. Even now EFT continues to make Einstein’s original
 339 mistake: He confuses global coordinate separations with measured quantities.

340 You too will probably find it difficult to avoid Einstein’s mistake.

**FIRST ADVICE
FOR THE ENTIRE
BOOK****FIRST STRONG ADVICE FOR THIS ENTIRE BOOK**

342 **To be safe, it is best to assume that global coordinate**
 343 **separations do not have any measured meaning. Use global**
 344 **coordinates only with the metric in hand to convert a**
 345 **mapmaker’s fantasy into a surveyor’s reality.**

346 Global coordinate systems come and go; wristwatch ticks and ruler lengths are
 347 forever!

Section 5.7 Local Measurement in a Room Using a Local Frame 5-11

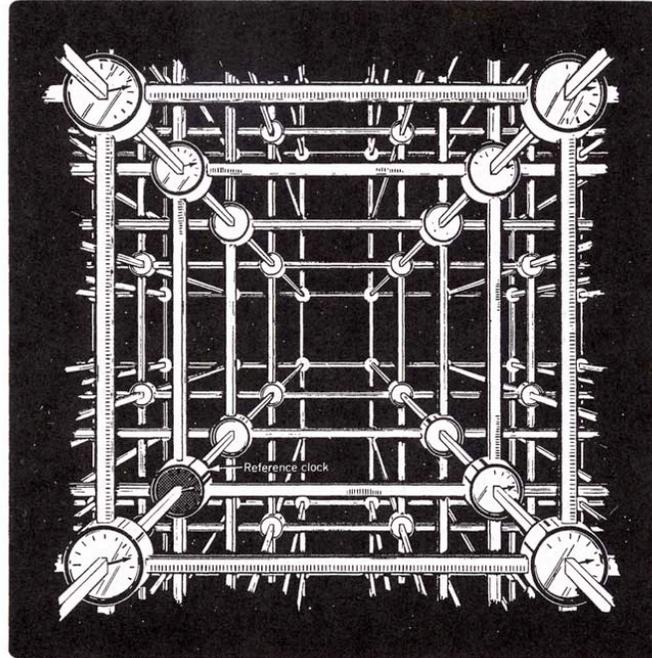


FIGURE 4 On a flat patch we build an inertial Cartesian latticework of meter sticks with synchronized clocks. This is an instrumented room (defined in Section 3.10), on which we impose a local coordinate system—a frame—limited in both space and time. Limited by what? Limited by the sensitivity to curvature of the measurement we want to carry out in that local inertial frame.

5.7 ■ LOCAL MEASUREMENT IN A ROOM USING A LOCAL FRAME

349 *Where we make real measurements*

350 *Of all theories ever conceived by physicists, general relativity*
 351 *has the simplest, most elegant geometric foundation. Three*
 352 *axioms: (1) there is a global metric; (2) the global metric is*
 353 *governed by the Einstein field equations; (3) all special*
 354 *relativistic laws of physics are valid in every local inertial*
 355 *frame, with its (local) flat-spacetime metric.*

356 —Misner, Thorne, and Wheeler (edited)

357 *No phenomenon is a physical phenomenon until it is an*
 358 *observed phenomenon.*

359 —John Archibald Wheeler

5-12 Chapter 5 Global and Local Metrics

Spacetime is
locally flat
almost everywhere.

360 Special relativity assumes that a measurement can take place throughout an
361 unlimited space and during an unlimited time. Spacetime curvature denies us
362 this scope, but general relativity takes advantage of the fact that almost
363 everywhere on a curved surface, space is locally flat; remember “flat Kansas”
364 in Figure 3, Section 2.2. Wherever spacetime is smooth—namely close to every
365 event except one on a singularity—general relativity permits us to approximate
366 the gently curving stage of spacetime with a local inertial frame. This section
367 sets up the command that we shout loudly everywhere in this book:

**SECOND ADVICE
FOR THE ENTIRE
BOOK**

SECOND STRONG ADVICE FOR THIS ENTIRE BOOK

**In this book we choose to make every measurement in a local
inertial frame, where special relativity rules.**

371 We ride in a *room*, a physical enclosure of fixed spatial dimensions (defined in
372 Section 3.10) in which we make our measurements, each measurement limited
373 in local time. We assume that the room is sufficiently small—and the duration
374 of our measurement sufficiently short—that these measurements can be
375 analyzed using special relativity. This assumption is correct on a *patch*.

Definition:
patch

DEFINITION 1. Patch

377 A **patch** is a spacetime region purposely limited in size and duration so
378 that curvature (tidal acceleration) does not noticeably affect a given
379 measurement.

380 *Important:* The definition of patch depends on the scope of the measurement
381 we wish to make. Different measurements require patches of different extent in
382 global coordinates. On this patch we lay out a local coordinate system, called
383 a *frame*.

Definition:
frame

DEFINITION 2. Frame

385 A **frame** is a local coordinate system of our choice installed onto a
386 spacetime patch. This local coordinate system is limited to that single
387 patch.

388 Among all possible local frames, we choose one that is inertial:

Definition:
inertial frame

DEFINITION 3. Inertial frame

389 An **inertial** or **free-fall frame** is a local coordinate system—typically
390 Cartesian spatial coordinates and readings on synchronized clocks
391 (Figure 4)—for which special relativity is valid. In this book we report
392 every measurement using a local inertial frame.
393

394 In general relativity every inertial frame is local, that is limited in spacetime
395 extent. Spacetime curvature precludes a global inertial frame.

396 Who makes all these measurements? The observer does:

Definition:
observer

DEFINITION 4. Observer = Inertial Observer

398 An **observer** is a person or machine that moves through spacetime
399 making measurements, each measurement limited to a local inertial
400 frame. Thus an observer moves through a series of local inertial frames.

Section 5.7 Local Measurement in a Room Using a Local Frame 5-13

Box 1. What moves?

A story—impossible to verify—recounts that at his trial by the Inquisition, after recanting his teaching that the Earth moves around the Sun, Galileo muttered under his breath, “Eppur si muove,” which means “And yet it moves.”

According to special and general relativity, what moves? We quickly eliminate coordinates, events, patches, frames, and spacetime itself:

- Coordinates do not move. Coordinates are number-labels that locate an event; it makes no sense to say that a coordinate number-label moves.
- An event does not move. An event is completely specified by coordinates; it makes no sense to say that an event moves.
- A flat patch does not move. A flat patch is a region of spacetime completely specified by a small, specific range of map coordinates; it makes no sense to say that a range of map coordinates moves.
- A local frame does not move. A frame is just a set of local coordinates—numbers—on a patch; it makes no sense to say that a set of local coordinates move.
- Spacetime does not move. *Spacetime* labels the arena in which events occur; it makes no sense to say that a label moves.

You cannot drop a frame. You cannot release a frame. You cannot accelerate a frame. It makes no sense to say that you

can even move a frame. You cannot carry a frame around, any more than you can move a postal zip code region by carrying its number around.

What does move? Stones and light flashes move; observers and rooms move. Whatever moves follows a worldline or worldtube through spacetime.

- A stone moves. Even a stone at rest in a shell frame moves on a worldline that changes global t -coordinate.
- A light flash moves; it follows a *null worldline* along which both r and ϕ can change, but $\Delta\tau = 0$.
- An observer moves. Basically the observer is an instrumented stone that makes measurements as it passes through local frames.
- A room moves. Basically a room is a large, hollow stone.

Why do almost all teachers and special relativity texts—including our own physics text *Spacetime Physics* and Chapter 1 of this book!—talk about “laboratory frame” and “rocket frame”? Because it is a tradition; it leads to no major confusion in special relativity. But when we specify a local rain frame in curved spacetime using (for example) a small range of Schwarzschild global coordinates t , r , and ϕ , then it makes no sense to say that this local rain frame—this range of global coordinates—moves. Stones move; coordinates do not.

401 The observer, riding in a room (Definition 3, Section 3.10), makes a sequence
402 of measurements as she passes through a series of local inertial frames. As it
403 passes through spacetime, the room drills out a *worldtube* (Definition 4,
404 Section 3.10). Figure 5 shows such a worldtube.



406 **Objection 1.** In Definition 4 you say that the observer moves through a
407 series of local inertial frames. But doesn't a shell observer stay in one local frame?



409 No! The shell observer is *not* stationary in the global t -coordinate, but
410 moves along a worldline (Figure 5). By definition, a local inertial frame
411 spans a given lapse of frame time Δt_{shell} , as well as a given frame volume
412 of space. In Figure 5 the first measurement takes place in Frame #1. When
413 the first measurement is over, global t/M has elapsed and the observer
414 leaves Frame #1. A second measurement takes place in Frame #2. The
415 range of r/M and ϕ global coordinates of Frame #2 may be the same as
416 in Frame #1. The shell observer makes a series of measurements, each measurement in a *different* local inertial frame.

5-14 Chapter 5 Global and Local Metrics

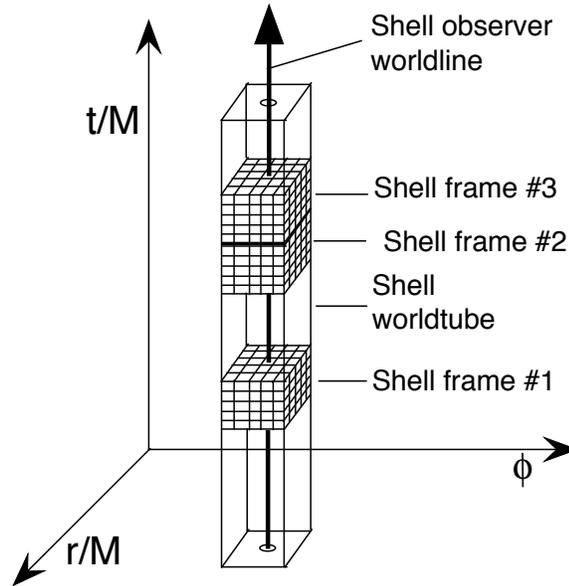


FIGURE 5 A shell worldtube (Section 3.10) that embraces three sample shell frames outside the event horizon. The shell observer carries out an experiment while passing through Frame #1 in the figure. He may then repeat the same experiment or carry out another one in Frames #2 and #3 at greater t coordinates. For simplicity each shell frame is shown as a cube. Each frame is *nailed* to a particular event at map coordinates $(\bar{t}/M, \bar{r}/M, \bar{\phi})$.

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Comment 1. Euclid’s curved space vs. Einstein’s curved spacetime

Figure 5 shows a case in which a shell observer stands at constant r and ϕ coordinates while he passes, with changing map t -coordinate, through a series of local frames, each frame defined over a range of r , ϕ , and t -coordinates. Figure 5 in Section 2.2 showed the Euclidean space analogy in which a traveler passes across a series of local flat maps on her way along the curved surface of Earth from Amsterdam to Vladivostok. Each of these flat maps is essentially a set of numbers: local space coordinates we set up for our own use. Similarly, each local frame of Figure 5 is just a set of numbers, local space and time coordinates we set up for our own use. A frame is not a room; a frame does not fall; a frame does not move; it is just a set of numbers—coordinates—that we use to report results of local measurements (Box 1). Figure 5 shows multiple shell frames, two of them adjacent in t -coordinate. Shell frames can also overlap, analogous to the overlap of adjacent local Euclidean maps in Figure 5, Section 2.2.



432

Objection 2. Whoa! Can a frame exist inside the event horizon?



433
434
435

Definitely. A frame is a set of coordinates—numbers! Numbers are not things; they can exist anywhere, even inside the event horizon. In contrast, the diver in her unpowered spaceship is a “thing.” Even inside the event

Section 5.7 Local Measurement in a Room Using a Local Frame **5-15**

436 horizon the she-thing continues to pass through a series of local frames.
 437 Inside the event horizon, however, she is doomed to continue to the
 438 singularity as her wristwatch ticks inevitably forward.

439 By definition, we use the flat-spacetime metric to analyze events in a local
 440 inertial frame. We write this metric for a local shell frame in a rather strange
 441 form which we then explain:

$$\Delta\tau^2 \approx \Delta t_{\text{shell}}^2 - \Delta y_{\text{shell}}^2 - \Delta x_{\text{shell}}^2 \quad (7)$$

Local flat spacetime
 → local inertial metric.

442 Choose the increment Δy_{shell} to be vertical (radially outward), and the Δx_{shell}
 443 increment to be horizontal (tangential along the shell).

444 Instead of an equal sign, equation (7) has an approximately equal sign.
 445 This is because near a black hole or elsewhere in our Universe there is always
 446 *some* spacetime curvature, so the equation cannot be exact. The upper case
 447 Delta, Δ , also has a different meaning in (7) than in special relativity. In
 448 special relativity (Section 1.10) we used Δ to emphasize that in flat spacetime
 449 the two events whose separation is described by (7) can be very far apart in
 450 space or time and their coordinate separations still satisfy (7) with an equals
 451 sign. In equation (7), however, both events must lie in the local frame within
 452 which the coordinate separations Δt_{shell} , Δy_{shell} , and Δx_{shell} are defined.

Connect global
 and local metrics

453 How do we connect local metric (7) to the Schwarzschild global metric? We
 454 do this by considering a local frame over which global coordinates t , r , and ϕ
 455 vary only a little. Small variation allows us to replace r with its average value
 456 \bar{r} over the patch and write the Schwarzschild metric in the approximate form:

$$\Delta\tau^2 \approx \left(1 - \frac{2M}{\bar{r}}\right) \Delta t^2 - \frac{\Delta r^2}{\left(1 - \frac{2M}{\bar{r}}\right)} - \bar{r}^2 \Delta\phi^2 \quad (\text{spacetime patch}) \quad (8)$$

457 Equation (8) is no longer global. The value of \bar{r} depends on *where* this patch is
 458 located, leading to a local wristwatch time lapse $\Delta\tau$ for a given change Δr .
 459 The value of \bar{r} also affects how much $\Delta\tau$ changes for a given change in Δt or
 460 $\Delta\phi$. Equation (8) is approximately correct only for limited ranges of Δt , Δr ,
 461 and $\Delta\phi$. In contrast to the differential global Schwarzschild metric, (8) has
 462 become a *local* metric. That is the bad news; now for some good news.

Local shell
 coordinates

463 Coefficients in (8) are now constants. So simply equate corresponding
 464 terms in the equations (8) and (7):

$$\Delta t_{\text{shell}} \equiv \left(1 - \frac{2M}{\bar{r}}\right)^{1/2} \Delta t \quad (9)$$

$$\Delta y_{\text{shell}} \equiv \left(1 - \frac{2M}{\bar{r}}\right)^{-1/2} \Delta r \quad (10)$$

$$\Delta x_{\text{shell}} \equiv \bar{r} \Delta\phi \quad (11)$$

465

5-16 Chapter 5 Global and Local Metrics



FIGURE 6 Flat triangular segments on the surface of a Buckminster Fuller geodesic dome. A single flat segment is the geometric analog of a locally flat patch in curved spacetime around a black hole; we add local coordinates to this patch to create a local frame. (Figure 3 in Section 3.3 shows a complete geodesic dome with six-sided segments.)

466 Substitutions (9), (10), and (11) turn approximate metric (8) into
 467 approximate metric (7), which is—approximately!—the metric for flat
 468 spacetime. *Payoff:* We can use special relativity analyze local measurements
 469 and observations in a shell frame near a black hole.



470 **Objection 3.** *What is the meaning of equations (9) through (11)? What do*
 471 *they accomplish? How do I use them?*



472 These equations are fundamental to our application of general relativity to
 473 Nature. On the left are measured quantities: Δt_{shell} is the measured shell
 474 time between two events, Δy_{shell} and Δx_{shell} are their measured
 475 separations in local space shell coordinates. These equations, plus the
 476 local metric (7) unleash special relativity to analyze local measurements in
 477 curved spacetime. In this book we choose to report every measurement
 478 using a local inertial frame.

479 **Comment 2. Left-handed ($\Delta y_{\text{shell}}, \Delta x_{\text{shell}}$) local space coordinates**

480 We find it convenient to have the local Δy_{shell} point along the outward global
 481 Schwarzschild r -coordinate and the local Δx_{shell} point along the direction of
 482 increasing angle $\Delta \phi$, on the $[r, \phi]$ slice through the center of the black hole. This
 483 earns the label **left-handed** for the space part of these local coordinates, which
 484 differs from most physics usage.

485 Figure 6 shows a geometric analogy to a local flat patch: the local flat
 486 plane segments on the curved exterior surface of a Buckminster Fuller geodesic
 487 dome.

Section 5.7 Local Measurement in a Room Using a Local Frame **5-17**

Summary:
local notation

488 We summarize here the new notation introduced in equation (7) and
489 equations (9) through (11):

- | | | |
|-----------|---|------|
| \approx | equality is not exact, due to residual curvature
and coordinate conversion (Section 5.8) | (12) |
| Δ | coordinate separation of two events within the local frame | (13) |
| \bar{r} | average r -coordinate across the patch | (14) |

490



491 **Objection 4.** *How large—in Δt_{shell} , Δy_{shell} , and Δx_{shell} —am I allowed*
492 *to make my local inertial frame? If you cannot tell me that, you have no*
493 *business talking about local inertial frames at all!*



494 You are right, but the answer depends on the measurement you want to
495 make. Some measurements are more sensitive than others to tidal
496 accelerations; each measurement sets its own limit on the maximum extent
497 of the local frame in order that it remain inertial for that measurement. If
498 the local frame is too extended in both the Δx_{shell} and Δy_{shell} directions
499 to be inertial, then it may be necessary to restrict the frame time Δt_{shell}
500 during which it is carried out. *Result:* Different measurements prevent us
501 from setting a universal, one-fits-all size for a local inertial frame. Sorry.



502 **Objection 5.** *What happens when we choose the size of the local frame*
503 *too great, so the frame is no longer inertial? How do we know when we*
504 *exceed this limit?*



505
506 There are two answers to these questions. The first is spacetime
507 curvature: Section 1.11 entitled Limits on Local Inertial Frames describes
508 this situation using Newtonian intuition. If two stones initially at rest near
509 Earth are separated radially, the stone nearer the center accelerates
510 downward at a faster rate. If two stones, initially at rest, are separated
511 tangentially, their accelerations do not point in the same directions, Figure
512 8, Section 1.11. These effects go under the name *tidal accelerations*,
513 because ocean tides on Earth result from differences in gravitational
514 attraction of Moon and Sun at different locations on Earth. If these tidal
515 accelerations exceed the achievable accuracy of an experiment, then the
516 local frame cannot be considered inertial.

517 The second answer to the question results from the global coordinate
518 system itself and the process by which the local inertial frame is derived
519 from it. This part is treated in Section 5.8.

5-18 Chapter 5 Global and Local Metrics

Box 2. Who cares about local inertial frames?

Sections 5.1 through 5.6 make no reference to local inertial frames. Nor are they necessary. The left side of the global metric predicts differentials $d\tau$ or $d\sigma$ (or $d\tau = d\sigma = 0$) between adjacent events. Of course we cannot measure differentials directly, because they are, by definition, vanishingly small. We need to integrate them; for example we integrate wristwatch time along the worldline of a stone. The resulting predictions are sufficient to analyze results of

any experiment or observation. No local inertial frames are required, and most general relativity texts do not use them.

Our approach in this book is different; we *choose* to predict, carry out, and report all measurements with respect to a local inertial frame. *Payoff*: In each local inertial frame we can unleash all the concepts and tools of special relativity, such as directly-measured space and time coordinate separations, measurable energy and momentum of a stone; Lorentz transformations between local inertial frames.

520 We may report local-frame measurements in the calculus limit, as we often
 521 do on Earth. For example, we record the motion of a light flash in our local
 522 inertial frame. Rewrite (7) as

$$\Delta\tau^2 \approx \Delta t_{\text{shell}}^2 - \Delta s_{\text{shell}}^2 \tag{15}$$

523 where Δs_{shell} is the distance between two events measured in the shell frame.
 524 Now let a light flash travel directly between the two events in our local frame.
 525 For light $\Delta\tau = 0$ and we write its speed (a positive quantity) as:

$$\left| \frac{\Delta s_{\text{shell}}}{\Delta t_{\text{shell}}} \right| \approx 1 \quad (\text{speed of light flash}) \tag{16}$$

Can take calculus limit in local frame.

526 We may want to know the instantaneous speed, which requires the calculus
 527 limit. In this process all increments shrink to differentials and $\bar{r} \rightarrow r$. For the
 528 light flash the result is:

$$v_{\text{shell}} \equiv \lim_{\Delta t_{\text{shell}} \rightarrow 0} \left| \frac{\Delta s_{\text{shell}}}{\Delta t_{\text{shell}}} \right| = 1 \quad (\text{instantaneous light flash speed}) \tag{17}$$

529 Equation (17) reassures us that the speed of light is exactly one when
 530 measured in a local shell frame at any r (outside the event horizon, where
 531 shells can be constructed). The measured speed of a stone is always less than
 532 unity:

$$v_{\text{shell}} \equiv \lim_{\Delta t_{\text{shell}} \rightarrow 0} \left| \frac{\Delta s_{\text{shell}}}{\Delta t_{\text{shell}}} \right| < 1 \quad (\text{instantaneous stone speed}) \tag{18}$$

5.8 ■ THE TROUBLE WITH COORDINATES

534 *Coordinates, as well as spacetime curvature, limit accuracy.*

Can use global metric exclusively.

535 We need global coordinates and cannot apply general relativity without them.
 536 Only global coordinates can connect widely separated local inertial frames in

Section 5.8 The Trouble with Coordinates 5-19

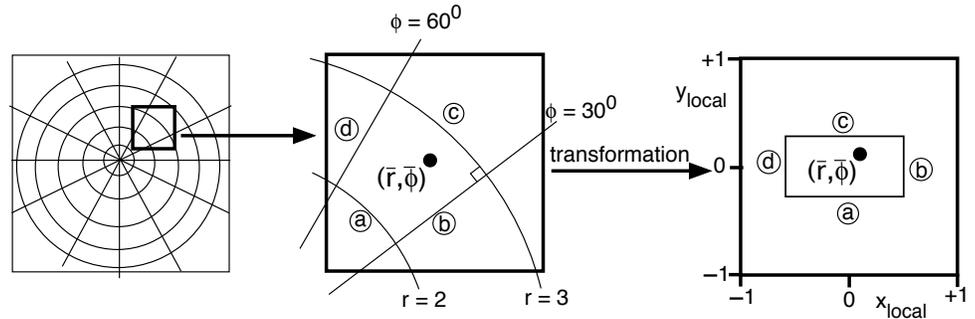


FIGURE 7 Inaccuracies due to polar coordinates on a flat sheet of paper. Coordinates in the middle frame are curved.

We choose to use local coordinates.

Approximation due to coordinate conversion

537 which we make measurements. Indeed, we can choose to use only global
 538 coordinates to apply general relativity (Box 2). Instead, in this book we *choose*
 539 to design and carry out measurements in a local inertial frame in order to
 540 unleash the power and simplicity of special relativity. In this process we fix
 541 average values of global coordinates to make constant the coefficients in the
 542 global metric. This allows us to write down the relation between global and
 543 local coordinates, equations (9) through (11), in order to generate a local flat
 544 spacetime metric (7).

545 But our choice has a cost that has nothing to do with spacetime
 546 curvature, illustrated by analogy to a flat geometric surface in Figure 7. The
 547 left frame shows polar coordinates laid out on the entire flat sheet. Choose a
 548 small area of the sheet (expanded in the second frame). That small area is, a
 549 *patch* (Definition 1) with a small section of *global* coordinates superimposed.
 550 This is a *frame* (Definition 2) whose local coordinate system is derived from
 551 global coordinates. The third frame shows Cartesian coordinates that cover
 552 the same patch, converting it to a local Cartesian frame, analogous to an
 553 inertial frame (Definition 3). What is the relation between the second frame
 554 and the third frame?

555 The exact differential separation between adjacent points is

$$ds^2 = dr^2 + r^2 d\phi^2 \tag{19}$$

556 In order to provide some “elbow room” to carry out local measurements on
 557 our small patch, we expand from differentials to small increments with the
 558 approximations:

$$\begin{aligned} \Delta s^2 &\approx \Delta r^2 + \bar{r}^2 \Delta \phi^2 \\ &\approx \Delta x^2 + \Delta y^2 \end{aligned} \tag{20}$$

Approximate due to (1) residual curvature plus (2) coordinate conversion.

559 Because of the average \bar{r} due to curved coordinates, equation (20) is not exact.
 560 The approximation of this result has nothing to do with curvature, since the
 561 surface in the left panel is flat. A similar inexactness haunts the relation

5-20 Chapter 5 Global and Local Metrics

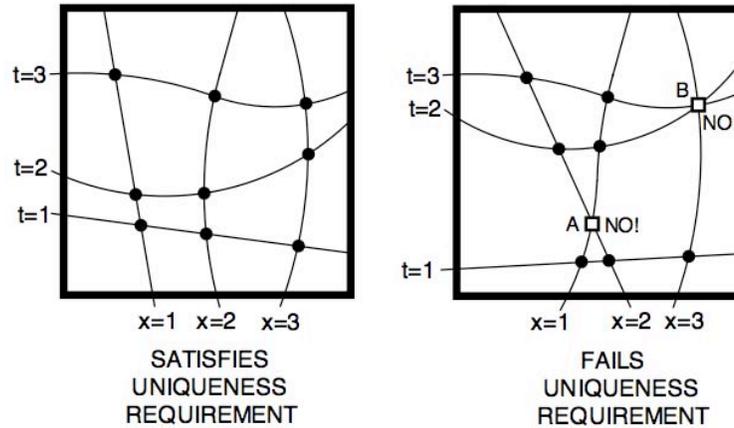


FIGURE 8 Left panel. Example of global coordinates that satisfy the uniqueness requirement: every event shown (filled circles) has a unique value of x and t . Right panel: Example of a global coordinate system that fails to satisfy the uniqueness requirement; Event A has two x -coordinates: $x = 1$ and $x = 2$; Event B has two t -coordinates: $t = 2$ and $t = 3$.

562 between global and local coordinates in equations (9) through (11). These
 563 equations are approximate for two reasons: (1) the residual curvature of
 564 spacetime across the local frame and (2) the conversion between global and
 565 local coordinates. In this book we emphasize the first of these, but the second
 566 is ever-present.

5.9 ■ REQUIREMENTS OF GLOBAL COORDINATE SYSTEMS

568 *Which coordinate systems can we use in a global metric?*

Some restrictions
on global coordinates

569 Thus far we have put no restrictions on global coordinate systems for global
 570 metrics in general relativity. The basic requirements are a global coordinate
 571 system that (a) uniquely specifies the spacetime location of every event, and
 572 (b) when submitted to Einstein's equations results in a global metric. Here are
 573 three technical requirements, quoted from advanced theory without proof.

FIRST REQUIREMENT: UNIQUENESS

Unique set of
coordinates
for each event

574 The global coordinate system must provide a unique set of coordinates for each
 575 separate event in the spacetime region under consideration.
 576

577 The uniqueness requirement seems reasonable. A set of global coordinates, for
 578 example t, r, ϕ , must allow us to distinguish any given event from every other
 579 event. That is, no two distinct events can have every global coordinate the
 580 same; nor can any given event be labelled by more than one set of coordinates.
 581 The left panel in Figure 8 shows an example of global coordinates that satisfy
 582 the uniqueness requirement; the right panel shows an example of global
 583 coordinates that fail this requirement.

Section 5.9 Requirements of Global Coordinate Systems **5-21**

Box 3. Find a particular local inertial frame.

How can we locate and label a particular local inertial frame on a shell around a black hole?

Ask a simpler question: How do we label and find one particular flat triangular surface on a Buckminster Fuller geodesic dome (Figure 6)? One way is simply to number each flat surface: triangle #523 next to triangle #524 next to triangle #525. Carry out this procedure for every flat triangle on the geodesic dome. The result is a huge catalog that we must consult to locate a given local flat segment on these nested Buckminster Fuller geodesic domes.

We could use a similar sequential numbering scheme to label and find a local inertial shell frame around a black hole,

sequential in both space and time. But we already have a simpler way to index a single local inertial frame:

Equations (9) through (11) provide a much simpler indexing scheme: the average values \bar{t} , \bar{r} , and $\bar{\phi}$. Average \bar{r} gives us the shell, average $\bar{\phi}$ locates the position of the local frame along the shell, and average \bar{t} tells us the global t -coordinate of the frame at that location—local in time as well as space. Three numbers, for example \bar{t} , \bar{r} , and $\bar{\phi}$, specify precisely the local inertial shell frame in spacetime surrounding a black hole.

584 In addition to the uniqueness requirement, we must be able to set up a
 585 local inertial frame everywhere around the black hole (except on its singularity.
 586 To allow this possibility, we add the second, smoothness requirement:

Smooth
 coordinates

587 **SECOND REQUIREMENT: SMOOTHNESS**

588 The coordinates must vary smoothly from event to neighboring event. In practice,
 589 this means there must be a differentiable coordinate transformation that takes
 590 the global metric to a local inertial metric (except on a physical singularity).

591 The third and final requirement seems obvious to us in everyday life but is
 592 often the troublemaker in curved spacetime.

Every event
 has coordinates.

593 **THIRD REQUIREMENT: COVERING OR EXTENSIBILITY**

594 Every event must have coordinates. Coordinates must cover all spacetime.

Good and
 bad coordinates

595 Coordinates that satisfy all three requirements we will call **good**
 596 **coordinates**. Coordinates that fail to satisfy all three coordinates we will call
 597 **bad coordinates**. In flat spacetime we can find *good* coordinates that satisfy
 598 all three requirements. *In curved spacetime there are frequently no good*
 599 *coordinates*.

Frequently:
 no good
 coordinates in
 curved spacetime

600 The third requirement is often the first to be violated, because in many
 601 curved spacetimes a single coordinate system cannot cover the entire
 602 spacetime while preserving the first two conditions. A simple example is the
 603 sphere, which requires two good coordinate systems because latitude and
 604 longitude coordinates violate the second requirement at the poles. We usually
 605 ignore this while using polar coordinates, even though these coordinates are
 606 bad at $r = 0$ (Box 3, Section 3.1).

607 **Comment 3. The (almost) complete freedom of general relativity**

608 There are an unlimited number of valid global coordinate systems that describe
 609 spacetime around a stable object such as a star, white dwarf, neutron star, or
 610 black hole (Box 3, Section 7.5). Who chooses which global coordinate system to
 611 use? We do!

5-22 Chapter 5 Global and Local Metrics

612 Near every event (except on a singularity) there are an unlimited number of
 613 possible local inertial frames in an unlimited number of relative motions. Who
 614 chooses the single local frame in which to carry out our next measurement? We
 615 do!

616 Nature has no interest whatsoever in which global coordinates we choose or
 617 how we derive from them the local inertial frames we employ to report our
 618 measurements and to check our predictions. Choices of global coordinates and
 619 local frames are (almost) completely free human decisions. Welcome to the wild
 620 permissiveness of general relativity!

5.10 ■ EXERCISES

622 5.1. Rotation of vertical

623 The inertial metric (7) assumes that the Δx_{shell} and Δy_{shell} are both
 624 straight-line separations that are perpendicular to one another. How many
 625 kilometers along a great circle must you walk before both the horizontal and
 626 vertical directions “turn” by one degree

- 627 A. on Earth.
 628 B. on the Moon (radius 1 737 kilometers).
 629 C. on the shell at map coordinate $r = 3M$ of a black hole of mass five
 630 times that of our Sun.

631 5.2. Time warping

632 In a given global coordinate system, two identical clocks stand at rest on
 633 different shells directly under one another, the lower clock at map coordinate
 634 r_L , the higher clock at map coordinate r_H . By *identical clocks* we mean that
 635 when the clocks are side by side the measured shell time between sequential
 636 ticks is the same for both. When placed on shells of different map radii, the
 637 measured time lapses between adjacent ticks are $\Delta t_{\text{shell H}}$ and $\Delta t_{\text{shell L}}$,
 638 respectively.

- 639 A. Find an expression for the fractional measured time difference f
 640 between the shell clocks, defined as:

$$f \equiv \frac{\Delta t_{\text{shell H}} - \Delta t_{\text{shell L}}}{\Delta t_{\text{shell L}}} \quad (21)$$

641 This expression should depend on only the map r -values of the two
 642 clocks and on the mass M of the center of attraction.

- 643 B. Fix r_L of the lower shell clock. For what higher r_H -value does the
 644 fraction f have the greatest magnitude? Derive the expression f_{max} for
 645 this maximum fractional magnitude.

Section 5.10 Exercises **5-23**

- 646 C. Evaluate the numerical value of the greatest magnitude f_{\max} from Item
647 B when r_L corresponds to the following cases:
- 648 (a) Earth's surface (numerical parameters inside front cover)
649 (b) Moon's surface (radius 1 737 kilometers, mass 5.45×10^{-5} meters)
650 (c) on the shell at $r_L = 3M$ of a black hole of mass $M = 5M_{\text{Sun}}$ (Find
651 the value of M_{Sun} inside front cover)
- 652 D. Find the higher map coordinate r_H at which the fractional difference in
653 clock rates is 10^{-10} for the cases in Item C.
- 654 E. For case (c) in item C, what is the directly-measured distance between
655 the shell clocks?
- 656 F. What is the value of f_{\max} in the limit $r_L \rightarrow 2M$? What is the value of f
657 in the limit $r_L \rightarrow 2M$ and $r_H = 2M(1 + \epsilon)$, where $0 < \epsilon \ll 1$. What
658 does this result say about the ability of a light flash to move outward
659 from the event horizon?
- 660 G. Which items in this exercise have different answers when the upper
661 clock and the lower clock do *not* lie on the same radial line, that is
662 when the upper clock is *not* directly above the lower clock?

663 **5.3. The International Space Station as a local inertial frame**

664 The International Space Station (ISS) orbits at an altitude of $d = 400$
665 kilometers above Earth's surface. Astronauts inside the ISS are (almost) in
666 free float, because the ISS approximates an inertial frame. It is approximate,
667 that is a *local* inertial frame because Earth's gravity causes tidal accelerations,
668 tiny differences in gravitational accelerations at different locations.

669 The size of the ISS along the radial direction is $h = 20$ meters. Inside the
670 ISS, at a point farthest from Earth, an astronaut releases a small wooden ball
671 from rest. Simultaneously in the local ISS frame, along the same radial line
672 but at a point nearest to Earth, another astronaut releases a small steel ball
673 from rest. If the ISS did not depart from the specifications for an inertial
674 frame, the two balls would remain at rest relative to each other.

- 675 A. Use a qualitative argument to show that tidal acceleration causes the
676 two balls to move *apart* in the local ISS frame.
- 677 B. Use Newtonian mechanics to show that in the local ISS frame the
678 wooden ball moves away from the steel ball with a relative acceleration
679 given by the equation:

$$a = \frac{2GM_E h}{(R_E + d)^3} \approx 5.1 \times 10^{-5} \text{ meter/second}^2 \quad (22)$$

680 Here the subscript E refers to Earth, and G is the universal
681 gravitational constant. How many seconds elapse in the ISS frame for
682 the distance between the two balls to increase by 1 centimeter?

5-24 Chapter 5 Global and Local Metrics683 **5.4. Diving inertial frame**

684 Think of a local inertial frame constructed in a free capsule that dives past a
685 local shell frame with local radial velocity v_{rel} measured by the shell observer.
686 Use Lorentz transformations from Chapter 1 to find expressions similar to
687 equations (9) through (11) that give coordinate increments Δt_{dive} , Δy_{dive} , and
688 Δx_{dive} between a pair of events in the diving frame in terms of \bar{r} , v_{rel} , and
689 global coordinate increments Δt , Δr , and $\Delta \phi$.

690 **5.5. Tangentially moving inertial frame**

691 Think of a local inertial frame constructed in a capsule that moves
692 instantaneously in a tangential direction with tangential speed v_{rel} measured
693 by the shell observer. Use Lorentz transformations from Chapter 1 to find
694 expressions similar to equations (9) through (11) that give coordinate
695 increments Δt_{tang} , Δy_{tang} , and Δx_{tang} between a pair of events in the
696 tangentially-moving frame in terms of \bar{r} , v_{rel} , and global coordinate increments
697 Δt , Δr , and $\Delta \phi$.

5.11. REFERENCES

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